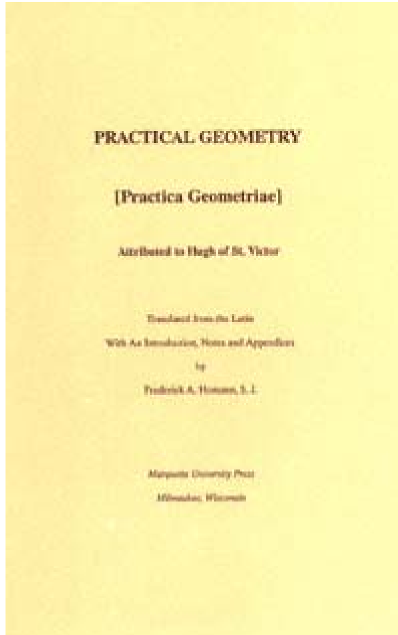

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Mediaeval Philosophical Texts in Translation

No. 29

Practical Geometry

[Practica Geometriae]

Attributed To Hugh of St. Victor

Translated from the Latin
With An Introduction, Notes and Appendices
by
Frederick A. Homann, S. J.

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To Alfred and Mimi
With Gratitude and Love

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Preface

This volume presents a translation of *Practica geometriae*, a twelfth century Latin treatise attributed by Abbé Roger Baron, its twentieth century editor, to Hugh of St. Victor. The translation is based on Baron's critical edition and is intended to provide accessible source reading in medieval Latin mathematics, education, and scholastic thought. A brief introduction considers the mathemat ical content, quality, and place of *Practica geometriae* in the history of mathematics without addressing textual problems.

The translation attempts to put the author's often labored, repetitive, and occasionally obscure medieval Latin into idiomatic English. It follows Baron's arrangement of the text into Prologue, Praenotanda, and three Chapters with his subdivisions of them into fifty-seven sections, identified, for example, as (#40). The author did not command a large technical Latin vocabulary; he knows but a few Greek terms from Euclid and Ptolemy. Indeed, he admits belatedly, in (#55), to using terms loosely. He uses the Arabic "*alidada*" once and replaces it with the medieval Latin "*mediclinium.*" His circumlocutions have been translated by current (and classical) English technical terms, for example, "*duo trianguli habentes eandem proportionem*" is rendered "two similar triangles." The Roman numerals of the manuscripts have been replaced by Arabic ones in the translation.

The medieval manuscripts Baron used have illustrations of varying quality, and some none at all. The figures in Baron's text,

themselves reproduced from an earlier edition of Curtze, have been redrawn, and others added, to interpret the translation. Figures modified, or added, are marked with an asterisk, e.g., Figure 19*. Illustrations from later sources have been supplied to suggest the persistence of the methods taught in *Practica geometriae*. To avoid distractions, numbered footnotes have not been used, but references for statements made in the introduction can be found, according to page number, in the notes.

Introduction

1. An Author and a Date

In 1955 the Abbé Roger Baron collated seven Latin manuscripts to establish a critical edition of a twelfth century treatise, *Practica geometriae* (Baron, 1955; 1966). A provisional text, done by Maximilian Curtze in 1897, had attracted historians of medieval mathematics, who, like Paul Tannery, saw *Practica geometriae* (PG) as an important witness to Latin mathematical practice and scholarship just before the twelfth century influx of Greek science into the West through Arabic channels (Curtze, 1897; Tannery, 1901).

On the basis of both internal and external evidence, Baron ascribed PG to Hugh (1096-1141), the distinguished theologian and master of the Abbey of St. Victor in Paris, an attribution widely accepted, even if not completely vindicated. PG is clearly related to Hugh's *Didascalicon*, which he set as a guide to the liberal arts, their nature, interrelations, and value for the sacred sciences (Taylor, 1961). Both works aim to preserve and integrate traditional learning. *Didascalicon* divided disciplines according to the theoretical and the practical. PG separated geometry into theoretical (speculative) and practical (applied) parts. *De arca Noe morali*, another of Hugh's treatises, took the Ark as a figure of the Church; it explained Genesis 6:15 with an account of the Ark's shape and dimensions and gave a table of linear measures: digit, foot, pace, and mile (Hugh of St. Victor, 1962). Its geometry and metrology, translated here in Appendix F, have counterparts in PG.

Medieval library catalogs listed PG with other treatises by Hugh of St. Victor, and one manuscript copy attributed PG to an otherwise unidentified Hugh (Baron, 1955).

Baron thought that PG was written just prior to *Didascalicon*, itself very likely composed in the late 1120s. Scholars also put *De arca Noe morali* in this period, but the dates and sequence of all three works are tentative (Baron, 1962).

This introduction will first provide an account setting PG within the complex history of practical geometry and applied technology in the Roman empire and its successors through the Renaissance. Then it will review PG itself for mathematical content and logical adequacy.

2. A Tradition of Practical Geometry

Practica geometriae (PG) can be securely located in the middle of a medieval Latin tradition of practical geometry. It stands as a point on a trajectory that started from technical concerns of the Graeco-Roman world and passed through an early period all but ignorant of Euclid. The trajectory intersected with Arabic scholarship in the latter haft of the twelfth century and matured to enrich Renaissance learning and aesthetics.

The tradition began with the Roman surveyors (*agrimensores*) who drew ideas from manuals of the Greek engineer, Heron, for boundary fixing, land measurement, map making, and urban design. Their legacy was preserved and enlarged by Western medieval monastic culture. "Our practical geometry is not something new," Hugh told his students in the Prologue to PG, "but only a collation of older, scattered material." In the century after Hugh, Fibonacci had fresh Latin translations of Euclid from Greek and Arabic. He knew the reasons behind the surveyors' methods. His practical geometry, more versatile than that of the early twelfth century, helped Leon Battista Alberti ground his theory and practice of the new perspective drawing. The surveyors' profession continued to flourish, and manuals such as those of Clavius (Clavius, 1606) and Love (Love, 1768) can be regarded as late products of the tradition.

To substantiate the claims of this précis, we shall mark key points on the trajectory with data derived from the histories of geometry, metrology, and architectural town planning. The history of medieval practical geometry was initiated by nineteenth century German and French scholars; after a period of dormancy due in

part to the disruption caused by World War I, it has flowered in recent decades, even if no comprehensive monograph yet exists and little account of it is found in general histories of mathematics. Synopses of four recent papers will together make a convenient base for our efforts to see PG as a significant middle point in the trajectory described above.

The most comprehensive study is Stephen K. Victor's preface to his critical editions of the late twelfth century practical geometries, *Artis cuiuslibet consummatio* and *Pratike de Geometrie* (Victor, 1979). To estimate the place of practical geometry in the Middle Ages, Victor analyzed its nature and its role in education, the changing relations between theory and practice, and the technology the geometry supported.

Victor noted that a book of practical geometry was simply a manual for measurement in one, two, or three dimensions. Physical instruments, or proportional numerical relations, or both, were needed to compute length, area, and volume. Practical geometry had a definite place in medieval schools. Treatises on the division of the sciences, Hugh's *Didascalicon*, for one, explained its role relative to other disciplines. For at least two of them, it served as a propaedeutic. Part of the foundations that philosophy and Scriptural exegesis each sought in the liberal arts was supplied by study of practical geometry. The already mentioned allegorical interpretation of the Ark's shape, in *De arca Noe morali*, thus appealed to geometry for support (Smalley, 1964, pp. 95-7).

In time, however, practical geometry assimilated some theoretical, or speculative, geometry. Awareness of Euclidean demonstration increased from the twelfth century onwards, as Greek and Arabic learning became commonly known in the schools. Indeed, as Victor remarks, after Fibonacci put Euclidean proofs in his text (c. 1220), theoretical validation of procedures became an explicit goal of practical geometry and often predominated, as we will observe later in Clavius's prolix treatise.

Victor suggests how the medieval texts taught about geometry (and, later, arithmetical counting) to a reader who wanted to learn, even if he would never use mathematics professionally in surveying, architecture, or town planning. *Pratike de Geometrie*, written in a Picardian vernacular, looked to a wide audience, including scholars as well as artisans. Merchants, exegetes, astronomers, and astrologers could all find in manuals like this something of interest.

Gillian R. Evans has culled theological and philosophical texts to isolate and analyze what she called the "sub-Euclidean" geometry of the earlier Middle Ages (Evans, 1976). Clagett had already noted that "the question of the translation into Latin of Euclid's *Elements* in the Middle Ages can be resolved into two principal sub-questions," i.e., that of the survival of older translations from the Greek (like the incomplete one ascribed to Boethius), and that of the making of fresh ones from Arabic or Greek, as Adelard of Bath did, in the twelfth century (Clagett, 1953). To these Evans added a third: that of the independent survival of a "sub-Euclidean" residue along side properly Euclidean material. Her search showed ample elements of geometry transmitted by Latin writers who did not know Euclid firsthand. Their simplified Euclidean material had a flavor other than that of *Elements* and often showed little understanding of the matter. Even so, their books had considerable impact on many readers whose primary interest was not geometry.

Several late Roman and early medieval sources exhibit this pattern, notably Macrobius, Martianus Capella, Cassiodorus, Boethius, Chalcidius, and Augustine. Like the later practical geometry authors, they transmitted geometric methods and ideas used early on with little or no help from the almost non-existent study of Euclid.

According to Evans, geometric ideas in Augustine and Boethius influenced Gerbert of Aurillac and Ailred of Rievaulx, though they put this legacy to different ends. The so-called *Geometria Gerberti*, was a surveyor's manual, adequate for practitioners, but not for geometers. Ailred had commonplace geometric images in his *De anima* to elucidate the incomposite nature of the soul. His use is superficial, but it did expect recognition from his readers. Similar appropriations to theology appeared in John Garland, Thierry of Chartres, Alan of Lille, and Hugh of St. Victor.

If medieval theologians drew on Augustine and the encyclopedists for geometric images, the practical geometry writers got "sub-Euclidean" fragments from the *agrimensores*, and Archimedean ideas, especially, from Vitruvius Rufus and Epaphroditus, whose defective texts will be reviewed later for Archimedean content as a legacy to the geometries of Gerbert and Hugh of St. Victor.

In a third study, Lon R. Shelby reviewed the place of geometry among the seven liberal arts (Shelby, 1983). Medieval university statutes on the teaching of mathematics are few and not very detailed. But noting how Hugh of St. Victor and Dominicus Gundissalinus, a twelfth century Spanish philosopher and translator who took directly from Arabic sources, distinguished between the theory and the practice of an art, Shelby identified three forms of

geometry in the High Middle Ages. Theoretical, or mathematical geometry, came to the West from the twelfth century onwards with translations of Euclid and Archimedes from the Arabic. In the Latin of Adelard of Bath and Campanus of Novara, Euclid became a subject of study in such faculties as offered lectures in all the seven liberal arts. Jordanus de Nemore and, at Oxford, Robert Grosseteste and Roger Bacon taught the new methods and applied them to problems in motion and light.

A second form was practical geometry. Gerbert and Hugh attest that this too was a lecture topic in the schools, as well as a working technique for the surveyor. Shelby calls a third form "constructive geometry." Medieval craftsmen, such as masons and architects, used geometrical forms to work the materials proper to their crafts. Carried by oral tradition and never a subject of formal study in the schools, constructive geometry was recorded in no scholastic treatise; rather it has to be

recovered from the artisans' work and their few books now left us. Perhaps the best known is the *Sketchbook* of the thirteenth century craftsman, Villard de Honnecourt (Bowie, 1959). In his imperfectly preserved notes, Villard began, "You will find good advice about the proper technique of masonry and the devices of carpentry, as well as the technique of drawing the forms just as the art of geometry (*ars de iometrie*) requires and teaches it." Roriczer's fourteenth century *Geometria deutsch* likewise taught how to make right angles, pentagons, hexagons, and octagons, but it never showed that these were correct procedures (Roriczer, 1486).

Shelby separated constructive from practical geometry. While neither had proofs, the former did not even have mathematical calculations, and it had a "mind set" quite different from that of computational practical geometry. He observed that master masons earlier were for the most part unlettered and outside the school tradition of practical and theoretical geometry (Shelby, 1972).

The fourth study to note is Nan B. Hahn's preface to the thirteenth century treatise *Quadrans vetus* and its principal source *Geometriae due sunt partes principales*, whose critical editions she produced (Hahn, 1982). The two texts taught the medieval surveyor how to use quadrant and astrolabe. These devices, like the modern navigator's sextant, measured angular altitudes of celestial bodies. Hahn detailed the measurement instruments of practical geometry: rods, right triangles, quadrants and astrolabes. Her essay complements the history of medieval geometry; it also supplements and brings up to date Gunther's standard history of the astrolabe (Gunther, 1932).

According to Hahn, Hugh's *Practica geometriae* anteceded both

texts and may be one of their sources. Their remote sources are found in Greek geometrical astronomy. Ptolemy's mid-second century *Planisphaerium* is the earliest extant treatment of the astrolabe. Late in the fourth century, Theon of Alexandria, a commentator and editor of Euclid, explained Ptolemy's work in a text that is now lost. Understanding of the nature and construction of the astrolabe passed next to the Arabs. The first Islamic scientific treatise on the astrolabe has been attributed to the eighth century Messahala (Nasr, 1976). Today his book is known only in John of Seville's mid-twelfth century Latin translation (which, in turn, shaped Chaucer's *Treatise on the Astrolabe*).

The quadrant's evolution can be traced, in part, from evidence in the tenth century pseudo-Gerbertian *Geometria incerti auctoris* (Bubnov, 1899), a practical geometry text Baron identified as a source for Hugh's *Practica geometriae*, along with the tenth century pseudo-Gerbertian

Liber de astrolabio (Baron, 1966). Hahn conjectured that these and later documents, including Sacrobosco's thirteenth century *Omnis scientia per instrumentum operative*, were sources for *Quadrans vetus* and *Geometriae due sunt partes principales*.

Hahn's essay shows that knowledge and use of practical geometry devices, and the astrolabe in particular, found in PG, were characteristic of the tradition prior to the incursion of Arabic scholarship in the West.

These four papers provide a matrix in which we now set key trajectory points chronologically in order to locate the position and significance of PG.

3. Chronology Table

300 BC
Euclid: *Elements*

250
Archimedes: *Measurement of the Circle*
Eratosthenes: *Measurement of the Earth*

20 AD
Vitruvius: *De architectura*

60?
Heron: *Metrica*

90
Columella: *De re rustica*

100 +
Corpus agrimensorum

150
Ptolemy of Alexandria: *Planisphaerium*

208
Forma urbis Romae Plan

400?
Marcus Iunius Nipsus: *Podismus*

430
Augustine

524
Boethius

529
Close of the Academy
Foundation of the Abbey of Monte Cassino

575
Cassiodorus Senator: *Institutiones*

820
The Plan of St. Gall

980?
Geometria Gerberti; Geometria incerti auctoris

1003
Gerbert (Sylvester II)

1125?
Hugonis practica geometriae

1141
Hugh of St. Victor

1142
Adelard of Bath: Latin Translation of Euclid

1190?
Artis cuiuslibet consummatio

1220
Fibonacci (Leonardo Pisano): *Practica geometriae*

1253
Robert Grosseteste

1275?
Quadrans vetus

1451
Leon Battista Alberti: *Ludi matematici*

1484
Matthew Roriczer: *Geometria deutsch*

1525
Albrecht Dürer: *Underweysung der Messung*

1604
Christopher Clavius: *Practica geometriae*

1768
John Love: *Geodaesia*

1785
Congress of Confederation: Land Ordinance

4. The *Agrimensores*

The rise of Latin practical geometry began with the Roman '*agrimensores*.' These field measurers were official civilian surveyors, whose military counterparts were '*metatores*.' Under the Roman emperors they functioned as a collegium. They had regular schools. Their main task was to measure unassigned state lands, and they were sometimes vested with judicial power. Early practitioners were '*gromatici*': those who used, or wrote about, the *groma*, a Roman device of two arms set at right angles, one for sighting, the other to set rectangular field direction (Dilke, 1987). When the *groma* was replaced by the *dioptra*, a multipurpose Greek instrument, '*gromatici*' became '*agrimensores*.' And they, rather than Greek geometers, set the pattern for an indigenous medieval practice that found an articulate twelfth century voice in Hugh of St. Victor.

Their records comprise the *Corpus agrimensorum*. Several

incomplete editions have appeared (Blume, Lachmann, and Rudorff, 1848-52; Cantor, 1875; Mortet and Tannery, 1896; Butzmann, 1970; Thulin, 1913 [1971]). The principal ancient witness is the illustrated sixth century *Wolfenbuttel codex arcerianus A*, though Bubnov argued to a more complete *Codex vetustissimus*, which, however, has yet to be located (Bubnov, 1899). Among the writers, who date from Domitian's time (81-96 AD) to the early fifth century, are Sextus Julius Frontinus (whose text was embedded in that of Agennius Urbicus), Siculus Flaccus, Hyginus Gromaticus, another Hyginus, Balbus, Marcus Iunius Nipsus, Epaphroditus, Vitruvius Rufus, and a pseudo-Boethius (Dilke, 1971).

To appreciate these texts relative to their medieval derivatives found in Boethius' *Geometria II* (Folkerts, 1970), Gerbert (Bubnov, 1899), Hugh of St. Victor (Baron, 1966), *Artis cuiuslibet consummatio* (Victor, 1979), *Quadrans vetus* (Hahn, 1982), Fibonacci's *Practica geometriae* (Boncompagni, 1862), and, in the Quattrocento, Alberti's *Ludi matematici* (Grayson, 1973), they must first be read in terms of their early contemporary, the Hellenistic engineer and manual author, Heron of Alexandria, who may be dated about 60 AD (van der Waerden, 1983, p. 181).

In 1896, R. Schoene found at Constantinople a manuscript copy of Heron's lost treatise *Metrica*. Its 1903 critical edition shed light on the sources of *Corpus agrimensorum* that Cantor, Lachmann, Rudorff, and other early interpreters did not enjoy (Heath, 1921, vol. II, p. 303). As examples will now show, procedures in "*Podismus*," the field measure notes of Nipsus, are simple rules for methods taken from *Metrica*. The anonymous and undated "*De jugeribus metiundis*" copied from Heron in an erratic way. The volume and area estimations of Archimedes, inaccessible to the *agrimensores* in his original account, were adapted by Heron for technical purposes and later impressed into a Roman practice that was derivative and crudely computational, yet remarkably accurate.

From these, and other fragments, such as Nipsus's *Fluminis varatio*, and the texts of Epaphroditus and Vitruvius Rufus, the medievals took, in haphazard ways, scraps of Archimedes' geometry of the circle, cylinder, and sphere (Clagett, 1978). The tenth century Gerbertian and pseudo-Gerbertian geometries that are PG's immediate sources, reproduced somewhat more Archimedean material than PG itself, which reported but a single formula: circle circumference = (22/7) diameter. Still Hugh got this, more likely, not from them, but from the Roman encyclopedia writer, Macrobius, whose texts he often criticized.

Our brief reading of Roman texts will show the Heron-Agrimensor-Medieval sequence in applied geometry.

5. Agrimensorial Geometry

The two principal extant texts of Nipsus speak the surveyor's language more than the geometer's. The first, "*Fluminis varatio*"

(literally, "the bending of a river"), addressed a standard task (Lachmann, 1848, p. 285). To measure the width of a river, use *groma* and measuring pole to construct two congruent right triangles. Make the span of the river the base of the first. The triangle congruent to it must have sides that can be measured directly. Then the measured width of its base is also the width of the river (Dilke, 1985, pp. 98-99). Nipsus did not validate this, the only field procedure described in his extant work, and later the diagram in *Codex Arcerianus* misinterpreted his scheme. Gerbert and Hugh gave better methods, with similar triangles to replace the unwieldy congruent figures the *agrimensores* seemed to use in day-to-day work.

Another text, *Podismus*, is diverse and didactic (Lachmann, 1848, pp. 295-302). Here Nipsus distinguished three measures: linear, planar, and solid. All this became common to the early medieval tradition, but Hugh was the first to specify altimetry, planimetry, and cosmimetry as basic parts of the genre of practical geometry. The standard angle is the right angle; the acute is less than it; the obtuse, greater. Nipsus set the volume of a box as the product of length, width, and height. The volume of a cylindrical cask was height times area of base, as in Heron. To compute the volume of a tapered circular cask was more difficult. Nipsus' process seems to measure it as the frustum of a cone by an averaging method of *Metrica* (Heron, 1903, II, 9), with Archimedean ratio 11/14 for p/4, through manuscript witness for the text here is defective.

Nipsus gave five ways to find lengths of various triangle lines. Each used the Pythagorean right triangle theorem that the square on the hypotenuse is equal to the sum of the squares on the sides. All of these are found in Heron, but here, as elsewhere, Nipsus chose examples that needed only integers. *Geometria incerti auctoris* later reproduced his sequence in order (Bubnov, 1899, pp. 338-41). PG did not consider these matters at all.

Right triangle examples have standard 3-4-5 Pythagorean triangles, or others that are integral multiples of these. The exercises about them that Heron and Nipsus proposed were not likely to be of immediate use: to find, for example, the sides of a right triangle, given its area and the length of its hypotenuse. Perhaps this was part of the *agrimensores*' general education, since there is no evidence that surveyors used triangulation procedures for land measurement before the time of Alberti.

Nipsus' first schemes treated triangle area in accord with whether the figure was right, acute, or obtuse. His last area formula was adapted from *Metrica I*, 8 and worked for any triangle (Thomas, 1957, vol II, pp. 470-77). Because he chose a 6-8-10 Pythagorean triangles as an example, he avoided the need for an old but intricate square root approximation procedure when he used Heron's remarkable formula, i.e.,

$$\text{Area} = R\sqrt{(s(s-a)(s-b)(s-c))},$$

where a, b, c are the triangle side lengths, and s = 1/2(a + b + c). So, for a = 6, b = 8, c = 10, he got s = 12, and area = 24. As usual, Nipsus gave directions for use, but no reasons for the directions. Heron, however, gave a proof of his formula and used examples where he had to approximate a square root to compute area. Al Biruni, an Arabic

witness, noted that the area formula was, in fact, the work of Archimedes [Dijksterhuis, 1987, pp. 412-13].

Geometria incerti auctoris (IV, 7) has a similar process for area (Bubnov, 1899, p. 341). But that text, *pace* Bubnov, did not copy Nipsus, though its process had the same effect. PG, which borrowed at length from *Geometria incerti auctoris*, might be expected to provide such area formulas. But it mentions only that for right triangles (area is one half base times width), and this in the Introduction, not the Planimetry chapter. Instead, Hugh confined his remarks to measurement of length, and seemed to expect his readers to know how to combine the border lengths that were measured, to get the enclosed field area. PG omitted all consideration of volume measurement.

The anonymous *De jugeribus metiundis* (*On Field Measurement*) listed some Roman measures (Lachmann, 1848, p. 355ff). A castral (military camp) acre (*juger*) is 288 perches, or 28,800 square feet. An acre is also 3 castral measures (*modia*). A plot (*tabula*) is 72 square perches. The text computed areas for squares, circles, and equilateral triangles. For circular measure, the value of p/4 was the Archimedean estimate 11/14; the value of square root of 3, needed for equilateral triangle area, was set as 2, a poor approximation for 1.728.

The area of a circular segment of diameter a and latitude (or arrow) b was

$$\text{Area} = 1/2(a+b)4+(1/14)(a/2)^2.$$

This distorted Heron's formula in *Metrica I*, 30:

$$\text{Area} = 1/2(a+b)b+(1/14)(a/2)^2.$$

Heron, however, used this only for circle segments less than a semicircle, and then only if a was not larger than 3b. [Heath, 1921, vol. II, p. 330]. But why the latitude b in Heron's statement was replaced with the constant 4 is not clear. No textual variants are recorded; perhaps it was a thoughtless imitation of Columella's example in *De re rustica* (V, ii), parallel to Heron's text.

The regular hexagon of side s has the area as the square of 6(1/3 + 1/10)s. Heron's *Geometria* had the square of (13/5)s, but "another book" has the first formulation, which, Heron said, is gotten from the equilateral triangle area, the square of (1/3 + 1/10)s, and is more accurate. (Heath noted that 6(1/3 + 1/10) = 13/5, and that Heron's

formulation in *Metrica I*, 17 of the square of the area of the hexagon as 27/4 times the fourth power of s is really the more accurate [Heath, 1921, vol. II, p. 327]. But the present procedure leads to no troublesome square roots, as might occur for Heron).

To get the area of an irregular (quadrilateral) field, the surveyor is to average the sides in pairs, and multiply them together. As usual, this invalid method is explained by an example, as are procedures for the area of a lune that seem to be modeled on those for triangle area.

Before we leave Roman times, we may note agrimensorial concerns in Pliny the Elder (79 AD), the architectural critic Vitruvius (c. 20 BC), and Columella, who wrote on agriculture (c. 60-65 AD).

For health and protection from storms, the Romans wanted to lay out towns and cultivated fields relative to prevailing winds. Geographical orientation had to be fixed. Pliny explained a crude method for this purpose, but accessible to rural folk with no technical equipment (*Natural History*, XVIII, 326). The noontime shadow mark and a furrow in the ground had to take the place of *groma* and surveyor's rod in his process.

Geometry first appears in *De architectura* when Vitruvius likewise considers the problem of the winds (I, vi.) Assuming that his readers knew how to divide a circle into sixteen equal parts, Vitruvius described how to make a "wind rose" related to the prevailing winds. This more accurate procedure Hyginus Gromaticus, in Trajan's time, preferred to that of Pliny, and he appended an alternative one that goes back to Alexandrian

scholarship (Dilke, 1971, pp. 57-8). Sundial design needed even more sophisticated methods. The site's equinoctial noontime shadow and meridian circle had to be determined, and an analemmatic figure drawn to compute daily hours for the time of the year. *De architectura* (IX, vii) explained, incompletely, the operative geometry and surveying techniques.

Orientation problems were more than technical ones for Roman surveyors, who were, in fact, successors of the Roman cultic augurs. Ritual aspects related the dividing lines of the Roman *templum* to the main terrestrial orientation. Temple symbolism, in turn grounded the standard practice for setting out the military camp lines that were as invariably rectangular as were new urban boundaries (Rykwert, 1988, p. 41ff.).

The task of orientation did not end with Roman cults and camps. Medieval church siting and cloister location had both symbolic and practical orientation aspects (Braunfels, 1972). Hugh's *Practica*

geometriae also needed a true north-south orientation of a gnomic dial to estimate mean apparent solar diameter and to compute solar altitude and the length of the solar orbit in Hugh's Ptolemaic cosmology.

Some geometric formulas we have already examined in Heron and the *agrimensores* are also found in the fifth book of Lucius Iunius Columella's agriculture treatise, *De re rustica* (Columella, 1941). To meet the request of Silvinus, a farmer friend, he listed area measurement rules for plane figures that agreed with those of Heron, notably for the equilateral triangle, the regular hexagon, and circular segments (*De re rustica*, V, ii). A detailed table of Roman field measures preceded the examples. But all this, Columella protested, was more the work of "geometers" than of country folk ("*id opus geometrarum magis esse quam rusticorum*"). One must learn to rely on the professional *mensores*, even as the architects do (*De re rustica*, V, i).

Columella did not name Heron among his sources, and it is possible Columella and Heron drew from a common source. Earlier, Columella cited a certain Mago, a trace of whose work (*ex libds Magonis et Vegoiae auctorum*) appears in the *Corpus agrimensorum* (Blume, Lachmann, Rudorff, 1846, p. 349). And Heron, as we know, referred to unidentified "more accurate writers" in connection with the circle segment formula (*Metrica* I, 31).

Columella, in any event, cannot be counted among the *agrimensores*, though he was indebted to them, and his technical work, along with theirs, was preserved for the medieval period and

appreciated in the Renaissance, not least of all by Leon Battista Alberti (Gadol, 1969, pp. 168, 203).

6. The Plan of St. Gall

Heron and the surveyor-engineers had a geometry that met technical needs from the decline of Rome to the rise of the Carolingian era. We may now inspect significant links between geometry and architecture in the latter period.

As a "science," Carolingian architecture worked according to geometric laws of ratio and proportion. By mid-twelfth century and Abbot Suger's abbey church of Saint-Denis, architecture was defined and practiced as applied geometry. Use of geometric proportions became at once technically and aesthetically necessary for the architect if the structure was to be stable and beautiful. Suger noted the use of (unidentified) "geometrical and arithmetical instruments" in building the

new abbey church (Panofsky, 1979, pp. 100, 239). Gundissalinus, Hugh's late contemporary, observed that every artist or craftsman could be found "acting as a geometer (*secundum geometriam practicans*)," making lines, surfaces, squares, globes, and all else in the materials proper to his art (Simson, 1974, p. 33).

The distance between the architect's scholastic knowledge of geometry and the craftsman's rudimentary practice reflected the fact that knowledge of geometry was largely the privilege of clerics literate in Latin. Libraries of monastic schools had copies of Vitruvius, and Mortet has indicated how architectural instruction was part of the geometry curriculum (Mortet, 1896; 1898). Knowledge of the liberal arts learned in cathedral or abbey schools like those of St. Victor or Corbie, put the architect in a class above his craftsmen. According to von Simson, it was "as scientists, not as practitioners, of geometry that the great Gothic architects had themselves depicted" in their memorials (Simson, 1974, p. 35).

A great Carolingian architect, and an heir to the agrimensorial legacy, was the ninth century ecclesiastic responsible for the "Plan of St. Gall." His paradigm for an ideal Benedictine monastic complex, now kept in the manuscript collection of the Library of St. Gall, evidences, in its subtle and systematic use of modular subdivisions, yet other resources of the tradition. Walter Horn's perceptive analysis of the Plan's scale and composition methods ended doubts that early medieval architects, ignorant of geometric similarity, drew to scale (Horn and Born, 1979).

The St. Gall plan that Horn studied is 44 (Carolingian) inches long by 30 inches wide; it is a copy of a now lost original. The draft contained some forty buildings in plan on a scale of 1:192. Horn

showed how 1/16 of a Carolingian inch was appropriate for representing each Carolingian foot on the ground. (16 units/inch x 12 inches/foot = 192 units/foot.) We may see this as a first link forward to a later English world that still uses scales that are multiples of 12.

Horn's analysis, in the view of A. H. Dupree, also effectively challenged a commonly accepted history of the measurement of length and of land division from the time of imperial Rome to that of Federal United States (Dupree, 1979).

Measurement history was assumed to resemble that of Western mathematics; after Archimedes, decline, and soon almost complete disorder in metrology, set in until modern science put order and quantitative precision in Western practice with a decimal metric system. Units such as inch (*uncia*: the twelfth part of a foot) and foot (*pes*: twelve

thumbs) found in Gerbert and Hugh and their successors, were embarrassing relics of disparate metrologies. Before Horn's work, Dupree noted, it would have been difficult to substantiate continuous use of a consistent measurement scheme and division grid from late Roman to early American times, even though similarity of land division patterns in Ohio and Kansas, for example, to Roman traces in the Po valley had been documented (Bradford, 1957). Linkage from Roman times, through the "Dark Ages," to early medieval England, was missing. Horn unearthed a link in the modules of the St. Gall planner. Carolingian times appeared as the system's center section. Continuity of the agrimensorial practice of plot division, Dupree concluded, now appeared unbroken, like the use of Latin, and a "single measuring system has survived within narrow limits of accuracy from Rome to America" (Dupree, 1979, p. 137).

The Roman system of plot division with a rectangular grid that the St. Gall architect later adapted is called 'centuriation.' ('*Centuratio*' and '*limitatio*' were equivalent technical terms in agrimensorial texts.) Subdivision was done by successive halving of the basic modules (Dilke, 1985, p. 88ff.). The standard module, or square 'century' had, in practice, an area of 200 Roman acres (*jugera*), or about 124.6 English acres. The "*jugerum*" represented a rectangle of 120 by 240 Roman feet, whose length was measured with the "*decempeda*," a surveyor's standard pole 10 Roman feet, or one perch, long.

Records of the size and shape of 'centuries' in several imperial Roman provinces have been preserved in the *Corpus agrimensorum* (Dilke, 1985, p. 91). Striking examples of

centuriation still remain in the Po Valley, in Campania around Capua, in Spain and southern France, in Dalmatia, and near Carthage in Tunisia. Archaeologists have also identified other likely centuriation patterns in England and the Low Countries. Good aerial photographs of many of them have been published (Bradford, 1957). Their ample evidence indicates an extensive and consistent surveyors' practice that reached to Carolingian times and after. That monastic houses like St. Gall fostered the tradition is supported by Ullman's evidence that "the gromatic and geometric capital of the world" was in the monastery school of Corbie, some 60 miles north of the abbey of St. Victor in Paris (Ullman, 1964).

The basic 2 1/2 foot module in the St. Gall Plan, Horn saw, came from the length of 40 feet by the continued halving that characterized agrimensorial plot division. This module generated the Carolingian measurement grid, whose significant units were the inch, the foot, and the step (2 1/2 feet), each related to conventionally chosen human

dimensions (inch=thumb, foot=12 thumbs). Significant groupings included the 40 foot side of the crossing square in the monastery church, the 160 foot supermodule of the Plan discerned in Horn's analysis, the Plan's 480 foot short side, and its 640 foot longer one.

Dupree lists some Roman sources for these units. The St. Gall "foot" is closer to that of Drusus (used in northwestern Roman provinces) than to the classical Roman foot. But once a unit length has been chosen, grid construction can proceed in the same way as one based on a slightly different length, provided that the name, here "foot," stays fixed. Dupree's analysis of the intervals in the Plan's grid showed that "while the units are clearly Roman, the changed social setting has seen a regrouping of the Roman units into a grid adapted to the life of a Benedictine monastery," rather than that of a Roman camp or fortified town.

A measuring system adapted for a Carolingian monastery has been identified. And so, says Dupree, "once the link between the Romans and the Carolingians is secured, the possibility of a link to the still living Anglo-American measuring systems requires the invention of no different language and no different social institutions from those we already know to have existed." For by 1300 English field measures had become part of statute law. This precluded any possibility of a break in continuity down to this day. Dupree sketches its progress from England to the American Congress of Confederation (1785) which chose, against a decimal system proposed by Jefferson, field measures based on the English mile. A mile square, 640 acres, was divided by continuous halving. Subdivision followed the pattern in the Plan of St. Gall. Roman *agrimensores*, the St. Gall planner, Gerbert, Hugh, thirteenth

century English and Federal American surveyors share a common tradition. All worked at measuring systems, and with the geometry that made these systems possible.

7. Gerbert's Geometry

Our next reference point is the first appearance of a substantial medieval Latin practical geometry. The *Geometria Gerberti* (Migne PL, 1853, v. cxxxix), ascribed to Gerbert of Aurillac, later Pope Sylvester II (d. 1003), was edited by N. Bubnov (Bubnov, 1899). He questioned the ascription, and separated an authentic *Geometria Gerberti* from the eighty-one chapters Migne had printed after it. The latter he termed *Geometria incerti auctoris*. They were, in his view, only miscellaneous propositions a tenth century editor collated and revised in polished Latin.

Some documentary sources he could identify. But to account for all the pseudo-Gerbertian material, plane and solid mensurational practice especially, Bubnov postulated the existence of a codex similar to *Codex Arcerianus*, but with more material from Nipsus and Epaphroditus than the texts earlier edited by Cantor (Cantor, 1875) and Mortet (Mortet and Tannery, 1896).

Geometria Gerberti and Geometria incerti auctoris represent, in their systematic use of similar triangles, significant progress beyond the *agrimensores*, whose records show use of congruent triangles only. The Plan of St. Gall was scaled and worked implicitly with similar figures, but we have extant no surveying manuals written between the sixth century *Codex Arcerianus* and the tenth century Gerbert texts to help us understand how similar triangle theory was recovered. Of course, use of the astrolabe, a Greek astronomical artifact developed by the Arabs and brought through Spain to Western scholars, implicitly required forming similar triangles with the alidade, an altitude-sighting device attached to its dorsal side. This may account for the emergence of similarity procedures in medieval surveying. Indeed, Hugh remarked that to cope with dimensions of the cosmos one had to make models or similar figures, and these Hugh took from the triangles fixed in his astrolabe readings (PG, #6).

Similarity theory, however, may not have been totally absent in agrimensorial practice. For Roman use of the concept of scale, and so of similarity theory, is found in "*Forma urbis Romae*," an early third century city plan incised in marble, and preserved in Rome's Palazzo Braschi (Dilke, 1985). The map's average scale is about 1:300. Some scale variation may be due, Dilke suggests, to careless

agrimensorial technicians. Scalar representation and, therefore, similarity theory were not unknown in imperial Roman technology and, arguably, came to ninth century monastic architects, to Gerbert and Hugh and their schools.

8. Hugh of St. Victor

Gerbert's work made possible Hugh's PG, whose date Baron set in the late 1120s, prior to Hugh's last great theological treatises. A later section will examine PG at length; here a few contextual remarks will suffice.

PG shows no concern for explanation in the Euclidean mode of the surveying procedures taught to "our students (*nostris*)." This suggests that translations of Euclid and Archimedes, that Adelard and others

were making, had not yet come to the author's attention and/or comprehension. But still unexplained is why Hugh did not incorporate all the Archimedean material found in Macrobius and the Gerbertian texts that he knew.

PG stands, in this way, as the end of that part of the practical geometry trajectory unable to benefit from firsthand acquaintance (at least in translation) with Archimedes and Euclid, and their Greek commentators. But PG does stand in debt to the new scholastic learning. The disciplined exposition in the three distinct but related parts of practical geometry is a refreshing development after the randomly arranged material in the Gerbertian texts. Like the still new *Summae*, PG's projects are taken in order, each developed from clearly marked starting points.

The use of Latin suggests that PG's readers were clerical scholastics. Exhortations to care in reading data from quadrant or astrolabe hint that they would have "hands-on" procedural experience and not just class lectures. *De cosmimetria's* inclusion of techniques to compute solar altitude and orbital circumference points to more speculative interests, though the seemingly unworkable schemes here raise doubt whether they were ever used. It is not unreasonable, then, to see PG as a last text whose readers were clerics literate in Latin. Vernacular texts appear soon after PG. The intellectual climate changed with the return of Greek science; social structures changed at the same time. Merchants and master craftsmen now needed geometry and arithmetic, but not in Latin (Murray, 1978, p. 203).

9. The Later Agrimensorial Tradition

The St. Gall architect adapted the Roman practice of centuriation for monastic community planning. Gerbert and Hugh established similar triangle surveyor procedures. Other developments next appeared in *Artis cuiuslibet consummatio, Pratike de geometrie*, and *Quadrans vetus* in the late twelfth century, as practical geometry books now also encompassed fractional arithmetic and astronomy, evidently for new uses and a wider audience. Fibonacci (Leonardo Pisano, 1180-1250) stands as a decisive figure in this stage (Vogel, 1971).

Fibonacci's *Practica geometriae*, written about 1220, had practical and theoretical roots (Boncompagni, 1862). It served both students of newly recovered Euclidean methods ("*demonstrationes geometricas*"), and those still engaged in traditional practice ("*qui secundum vulgarem*

consuetudinem, quasi laicali more, in dimensionibus voluerint operari"). Though he accorded Euclidean demonstrations priority, Fibonacci taught "*agrimensores*" how to measure triangle area with knotted cords and how to use flexible reeds to arc length. His Archimedean legacy came from Heron, but now by way of the Arabs (Clagett, 1978, p. 199).

Fibonacci mastered the *Liber embadorum*, which Plato of Tivoli, about 1145, had translated from the Hebrew geometry of Savasorda (Abraham bar Hiyya), itself based on Arabic scholarship. He also borrowed from the Latin *Verba filiorum* that Gerard of Cremona had rendered from a ninth century Arabic text done by the Banu Musa ibn Shakir, the "Three Brothers," each a respected mathematician. In the Roman tradition, Fibonacci recorded the diverse and intricate field measures of his native Pisa, just as Gerbert and Hugh (in *De arca Noe morali*) took care to explain current field measures to their students. Rich, diverse, and confident, Fibonacci's geometry collated work of earlier scholars, but, like Hugh's PG, was not original. His algebra, however, was innovative and earned Fibonacci a permanent place in the history of mathematics (Boyer, 1968, p. 278ff).

Medieval applied geometry also helped Leon Battista Alberti (1404-1472) to develop pictorial perspective. The link between a perspectival theory of vision (as found in Hugh's PG) and Brunelleschi's rudimentary perspectival technique came from understanding a picture as a cross section of the pyramid of sight. Alberti set the problem of planar depiction as one of applied plane geometry. According to Gadol, Alberti's construction rules came from the connection he saw between the intersecting picture plane

and the intersecting measuring rod of the surveyor when the rod is taken as an infinitely narrow plane intersecting the sight lines to the object: Surveying and perspective represent two modes of seeing: seeing when measuring a three dimensional object at a certain distance from the viewer, and seeing a three dimensional object through a "glass" on which its outlines can be traced. The affinity between the two consists in the fact that the surveyor's sighting, the first mode of seeing, may be interpreted as a special case of the second one. The surveyor's measuring rod, and the sightings obtained on it, present, in fact, a profile, or one cross section of a visual pyramid when thus interpreted. (Gadol, 1969, p. 40)

Alberti transposed these surveyors' sightings to the cross section of the visual pyramid on the picture plane. This let him work out the projection rules he set in *De pictura*/*Della pittura* for pictorial perspective. Surveyors and painters generate triangles, and lengths measured by visual rays may be represented proportionally on the

picture plane, just as they are projected on the surveyor's rod. Thanks to Alberti, practical geometry in part became perspectival geometry. Later, Pascal (1640), Desargues (1648), and Poncelet (1818) elevated this to the completely theoretical level of projective geometry, a far remove from earthy agrimensorial roots (Boyer, 1968). Agrimensorial interests, nonetheless, were a long part of Alberti's life.

Between 1431 and 1434, and prior to *De pictura* and its Italian counterpart *Della pittura*, Alberti composed *Descriptio urbis Romae*. To explain triangulation surveying, he included a table of sightings he made for Roman monuments. He described a device he designed to get proportionate widths of an object seen at various distances. It could also compute the distance of the object, given its width, or its width, given the distance, by means of similar triangles since proportionate quantities varied according to the distance of the object. Surveying theory became at the same time his key to perspectival representation. Alberti's procedure here was like that of determining the width of a river by rod and similar triangles. Details are given later in *Ludi matematici* (c. 1450), but we may recognize the problem as that found in the *agrimensores*, Gerbert, and Hugh. *Ludi matematici* also explained the same method Gerbert and Hugh had for the height of a tower when the distance between the tower and observer is known (Grayson, 1973). Indeed, every exercise Alberti proposed in his book for Meliaduso d'Este, the new abbot of Pomposa, looked to management problems of a large estate, land reclamation, and crop or field measurement. It is a fifteenth century addition to the *Corpus agrimensorum*. Alberti admitted his debt to the "ancients" Columella and Savasorda, as well as to the "modern" Fibonacci.

Book I of *De pictura*/*Della pittura* defined geometrical terms, and explained such Euclidean ideas as proportion and similar figures, despite Alberti's disclaimer that he spoke as painter, not as mathematician. He named no sources for his canons of perspective, but several parts of his Latin text closely parallel Fibonacci's *Practica geometriae*, a book written, as we know, for both "those who would work according to geometric demonstrations, and those who would follow common or, as it were, lay practice." (Edgerton, 1975, p. 80) A fifteenth century Italian translation of Fibonacci was, in any event, available.

Of course, in addition to the surveyor's manuals and Fibonacci, the geometric optics of Alhazen, Peckham, Witelo, and Bacon enriched the resources at Alberti's call. Latin and Arabo-Greek traditions together empowered the Florentine humanist.

Alberti was an early figure in the Italian Renaissance of mathematics. His perspective theory and triangulation surveying were original contributions at the time when mathematicians were repossessing the geometry of Euclid and Archimedes and developing the algebra inherited from the Arabs. Gutenberg's invention of printing with movable type assured wide dissemination of their work and readily available fresh editions of the ancients. The program proposed by Johann Mueller (Regiomontanus) in 1474, but precluded by his untimely death in 1476, included texts of Euclid, Apollonius, Archimedes, Heron, Ptolemy, Jordanus, Witelo, as well as his own important trigonometry text, *De triangulis omnimodis*.

Practical geometry texts, especially for surveyors, appeared both in the vernacular (often Italian) and in Latin. Among them are Cosimo Bartoli's *Del Modo di misurare* (Venice, 1589), Giovanni Pomodoro's *La Geometria prattica* (Rome, 1624), the *De quadrante geometrico* (Nuremberg, 1594), usually referred to Cornelius de Judeis, but really written by Levinus Hulsius, and the *Geometria practica* (Rome, 1604) by Christopher Clavius (Smith 1923).

As a last key item for study of the agrimensorial trajectory, we may pair the seventeenth century Latin *Geometria practica* of Clavius with John Love's eighteenth century English manual *Geodesia: or, The Art of Measuring Land* (Love, 1768). Clavius wrote just before Napier published his discovery of logarithms (1614); Love used them in decimal form all throughout his work. Logarithms and decimal notation facilitated the surveyor's computations even more than the slow replacement of Roman with Arabic numerals did.

Love is a step beyond Clavius, even as Fibonacci surpassed Hugh; his manual, in its many editions, was part of the tradition that would form Colonial American practitioners such as George Washington, Thomas Jefferson, George Mason, and David Rittenhouse. He, however, did not share Clavius's concern with the underlying theory.

Clavius published *Geometria practica* toward the end of his long tenure (1564-1612) as professor of mathematics at the Jesuits' Collegio Romano. Like his earlier *Astrolabium* (Rome, 1593), this treatise explained the use of measuring instruments and, in addition, the theory behind them. The text is divided into eight books. After an introductory analysis of trigonometric ideas, Clavius told how to use different types of quadrants to compute lengths and angles (Books 2 and 3). Books 4 and 5 gave formulas for area and volume and finished with the remark that "with these five sections, all three parts of practical geometry, as we

understand it, have been explained." There follow sections on Geodesy, Isoperimetry, and diverse other topics.

Clavius here recorded his own progress on long time unsolved problems, notably, to supply a proof of the plane isoperimetric theorem (i.e., the circle alone of all plane figures with the same perimeter length has the greatest area) and a way to effect the quadrature of the circle (i.e., to construct a square equal in area to a circle with a given radius).

In the course of his argument, Clavius evaluated the work of several predecessors and contemporaries, among them Albrecht Dürer and Joseph Scaliger. Dürer's *Underweysung der Messung mit dem Zirkel und Richtscheyt* (1525) was itself a practical geometry for artisans and builders. Very few mathematicians in the sixteenth and seventeenth centuries paid any attention to his constructive procedures and experimental projective geometry, which were indigenous to the practical, not the speculative tradition. Clavius, who bridged the two, demonstrated how some of Dürer's widely used construction rules for linear figures, such as the regular pentagon, gave close approximations rather than exact results (Book VIII, Prop. 29, Th. 11). Earlier, in the *Commentary on Euclid's Elements* (1574), he observed that "Albert (Dürer) supplied no reasons for his procedure (*Huius praxis nullam affert rationem*)." Indeed, Dürer's manual was close in tone to Villard De Honnecourt's *Sketchbook* in its patterns and directives.

In contrast, Clavius's *Geometria practica* drew heavily on the Greek geometers Pappus, Theon, Heron, and Zenodorus, all recently available in Latin translations, as well as Euclid, Apollonius, and Archimedes, and put their resources to surveying

and astronomical mensuration. His project was in the agrimensorial tradition, but its execution much in debt to the theoretical tradition, as this was recovered by the Latin mathematicians and humanists.

Prior to a description, in Book VII, of his work on the circle quadrature problem, Clavius reviewed efforts by Campanus of Novara (1296), Nicholas of Cusa (1464), Orontius Finaeus (1555), and the contemporary Joseph Scaliger (1540-1609). Scaliger's *Cyclometrica elementa* (1594), Clavius showed, had adopted in clumsy fashion Arabic interpretations of Archimedes and even misunderstood Archimedes' fundamental circle measurement theorem. He then proceeded to give a constructive solution to the quadrature problem that had theoretical basis, but remained, literally, a physical construct. It became a genuine part of practical geometry. Like isoperimetry, the circle quadrature problem found a natural place in the traditional sequence of linear, area, and volume measurement that Hugh had specified for the genre some

five centuries earlier. Clavius's work is yet another and late key point on the agrimensorial trajectory.

Continuity and change in the practical geometry tradition mark the many books that incorporate arithmetic just beyond the ken of Clavius. Logarithms first appeared in 1614 in John Napier's *Mirifici logarithmorum canonis*, and their value for the astronomer and surveyor was recognized at once. John Love's *Geodaesia* stands as a surveyor's text that has assimilated this new device into traditional practice still divided along the lines set by Hugh of St. Victor and brought to maturity by Clavius (Love, 1768).

Love, to start, taught the process of computing square roots with logarithms. Subsequent chapters explained how to construct various plane figures, and what are the basic field measures. Chapter Five treated measuring rods and instruments, including now the magnetic compass. After that, the text taught "how to apply all the foregoing matters together, in the practical surveying of any field, . . . by various instruments; how to cast up the contents of any plot of land, to lay out new lands, and how to reduce and divide lands." A trigonometry chapter led to another that treated "of heights and distances, including . . . how to make a map of a river or harbour" (Love, 1768, pp. 146-96).

Surveying, Love claimed, was first raised to an art by the Egyptians, who held its practitioners in high honor. Indeed, "the usefulness, as well as the pleasant and delightful study, and wholesome exercise thereof, tempted many to apply themselves thereto." The Romans so esteemed this knowledge, that they judged "that man to be incapable of commanding a legion, that did not possess at least so much geometry, as to know how to measure

a field." So much is it a pleasant study and innocent exercise, in his view, that no one who has entered into the study of geometry or geodesy "can ever after wholly lay it aside." But, he remembered, his goal here is to be practical and to teach young surveyors, especially in an isolated America, easy and reliable methods to lay out new lands.

Six centuries separate Hugh of St. Victor from John Love. Six centuries separate the last of the Roman *agrimensores* from Hugh. All had common tasks, and could have appreciated one another's work, even if Love, Clavius, and Alberti had conceptual devices unknown to Hugh and his predecessors, just as Hugh and Gerbert enjoyed ideas beyond Nipsus and Epaphroditus. (Heron, perhaps, knew more than any of them).

With these considerations, we may see Hugh of St. Victor's *Practica geometriae* as a center point in a trajectory that linked imperial Rome and Federal America. The agrimensorial tradition worked at perennial tasks of technology in service to agriculture and urban development. The technology, of necessity, was rooted in a theory that could support practical surveying and Neoplatonic speculation. The art of land measurement belonged equally to the unschooled master craftsman and to the scholastic cleric. Its history should be written with the histories of mathematics, the university liberal arts curriculum, and architecture. Hugh sustained the tradition. PG has no mathematics not known to the Greek tradition, but its structure and thematic development could prepare the reader for the more exacting study of Euclid. Procedures received in haphazard fashion were here classified. Here, too, as in his propaedeutic and theological work, Hugh allowed his indebtedness to his predecessors and raised the reader's historical consciousness.

Little explicit citation of Hugh's *Practica geometriae* text by subsequent writers has been identified up to now, though, as noted, Victor and Hahn have good reason for thinking that PG was a source for the practical geometry texts that they edited. Another witness, a thirteenth century practical geometry text kept in the Bodleian Library (BL Digby 166), reproduced part of PG with a preface by the unknown author (Victor, 1979, p. 39). So it is, rather, Hugh's division of an art into "*pars speculativa*" and "*pars practica*," along with the altimetry-planimetry-cosmimetry configuration, that marks his original contribution to the legacy given him from the Roman *agrimensores*. Hugh handed on more than he had received; his academic heirs did him the honor of imitation.

10. A Topical Outline of Practica geometriae

Preface: Purpose and tribute to predecessors.

Praenotanda (#1-6): Definitions; division of geometry into theoretical and practical; specific goals; geometry of similar right triangles; models of the celestial sphere and its great circles.

I. Altimetry (#7-35): Triangles and circles in measurement problems, and their geometry. The astrolabe and its quadrant face. Isoplane height problems for objects on the horizon and for nearby objects. Four ways to measure height when the triangle base is known; three ways when the base is not known. Two-station methods. Other instruments and their

use: triangles, rods, and mirrors. Heteroplane techniques. Depth measurement: visual techniques, mechanical devices.

II. Planimetry (#36-38): Three astrolabe techniques to measure level lengths.

III. Cosmimetry (#39-57): Introduction: Earth as a center point in the cosmos. Ancient received values for the diameter and circumference of the earth. Altitude of the sun computed in Egypt and elsewhere; the diameter and circumference of the solar orbit. The diameter of the sun and the length of the earth's shadow. Lemmas for geometric optics, horizon and vision problems.

11. Commentary on the Text

The graceful Latin of PG's introduction sets the goal of preserving and collating the work of unnamed predecessors, whose scholarship is to be honored if not surpassed. Though the students ("*nostris*" = "ours") are not identified, the use of Latin suggests schoolmen, while the content suggests surveyors and technicians. In contrast to the keen historical interest of *Didascalicon*, PG nowhere mentions Euclid or Boethius, though Baron's historical apparatus identified their geometries, along with those of Gerbert, as sources for PG. PG, however, does credit Pythagoras (#21) and Eratosthenes (#39) with specific results. But Macrobius, the fifth century Roman encyclopedia writer, also gets repeated criticism for deficient astronomical computations (#40, 45, 57).

Practica geometriae, here translated as "practical geometry," has three parts: altimetry, planimetry, and cosmimetry. This division, albeit in expanded and altered form persisted in the genre well into the seventeenth century. *Didascalicon* put it this way: Planimetry

measures the plane, i.e., the long and the broad, altimetry heights and depths, and cosmimetry things spherical and volumetric (Taylor, 1961, p. 70).

Euclidean definitions that pseudo-Boethius and Gerbert reported, contrast with those in PG. Euclid defined a (geometrical) point as "that which has no parts," and a line as "length without breadth," so that "the terminus of a line segment is a point." PG's points seem more substantial: "A point has the power to let a line emerge from it in any direction, and to come to it from any direction (#1)." Lines and circumferences enjoy similar powers. Surfaces are made by adjoining lines to form a plane boundary, solids by adjoining surfaces to form the boundary of a volume. Plane triangles can be rectilinear or curvilinear,

e.g., a right triangle whose (linear) hypotenuse, the "*anabibazon*," is a quadrant arc, as the geometry of the Ptolemaic world sphere needed (#3-6).

Area theory is rudimentary; rectangle area is base times width, and the area of the right triangle on either side of the diagonal is half the rectangle area. The ratio of circumference to circle diameter is the Archimedean estimate of three and one seventh (#40). PG says nothing of circle area. Division of the Ptolemaic world sphere into equal octants whose boundaries are curvilinear triangles is the only occurrence of solid geometry (#3-5). In contrast, *Geometria Gerberti* treats (unevenly) many more surface area and solid volume measurements, as do the later *Artis cuiuslibet consummatio* and *Ouadrans vetus*. The Pythagorean triangle theorem appears in *De arca Noe morali* (see Appendix F) and in *Geometria incerti auctoris* (see Appendix C), but not in PG, despite Baron's citation (Baron, 1966, p. 19).

Altimetry (#7-35) is the most applicable and mathematically most adequate section. Borrowing at length from Gerbertian sources, it builds systematically on the basic right-triangle theory stated in the preface and often reiterated. Indeed, the repetition suggests that PG may be classroom notes, perhaps organized and approved by Hugh, and with his own introduction. One can imagine him telling students: "This is an astrolabe. Here is how to use it . . . ." He warns of misapplications, and common student errors. There are bits of professional humor, infectious enthusiasm, and serene confidence in the power of reason (#28).

How can one measure the height (*altitudo*) of a castle or a mountain? If the surveyor (*mensor*) knows the distance D of the

object (*altitudo metienda*) from himself along a level surface, and the ratio R of the height H to distance D, then H is known: H = RD. The astrolabe, gotten from the Arabs and introduced to the West by Gerbert, is the most useful device for computing R. PG uses only its quadrant, or dorsal, face, where three contiguous squares, whose sides each have twelve divisions, are set about a center to which an alidade, or sighting device, is fixed. The surveyor adjusts the alidade until he can see the top of the object through its two apertures. The alidade reading of height to base is then the ratio of the triangle made by the object and the viewer.

To measure D directly may not be possible, so PG gives several forms of a "two-station" method to resolve the difficulty. (See Appendix A for details). Cosmimetry (#42) tried to use the method to compute solar altitude.

Heteroplane problems occur when the surveyor, standing on a plain, has to determine a height set above on a mountain, or, when standing on a hill, he must determine one set below in a valley (#29 *et seq*.). PG, repeating *Geometria Gerberti*, explains a "two-station" method with a quadrant in the customary way with an example rather than by Euclidean theory.

Other altimetry methods employ mirrors or water reflectors (#27-28). They apply Heron's theorem that the angle of light incidence on the mirror equals the angle of its reflection, to get similar triangles with measurable sides and ratios.

To measure the depth of a well, for example, a plumb bob and cord will often suffice. Similar triangle methods can also be used (#32-34). Illustrations in practical geometry manuals of the Renaissance testify that the methods stayed essentially unchanged (Smith, 1923, vol. II, p. 346).

Altimetry closes with a curious, incompletely described way to measure the depth of a lake or river (#35). A weighted float device is made that sinks to the bottom of the pool, and, on striking bottom, releases the float. Appendix B proposes a reconstruction of the device. The time taken for the device to sink to the bottom and refloat is carefully measured. If the time is found for a known depth, it is easy to calibrate the device for all others. Baron's reference to *Geometria incerti auctoris* III, 24 (Bubnov, 1899, pp. 333-34) seems misdirected. A better reference is to III, 26, whose translation is here given in Appendix C. Here the "*incertus auctor*" adds a bit of professorial humor. Tired minds need relief, he says, and a whimsical use of the Pythagorean theorem for a bow-and-arrow experiment will refresh the spirit. "*Incertus auctor*" tips his

hand. Hugh does not. Altimetry ends with the casual remark that he knows a way some people measure lake or river depth. He hasn't tried it, but he should mention it. The dullest student can see that it is wildly impractical, a humorous finish to an often tedious lecture, a rare bit of humor in a mathematical text.

Alberti later suggested the same idea in *Ludi matematici*, and thought it might work if there were no ocean currents (Grayson, 1973, p. 145). He got it from "*scrittori antichi*," perhaps from PG, or from one of PG's Latin sources noted in Appendix B.

Planimetry (#36-38) applies similar triangle methods to planar length measurement. Did Hugh know Gerbert's sometimes erroneous methods for areas of equilateral and isosceles triangles and regular polygons? Or the circle area formula? One looks for directions how to compute areas of plane figures, but they are not here. PG warns against

common procedural mistakes. Similarity theory rules are repeated, as though to clinch the techniques that replaced the unwieldy congruence methods of the *agrimensores*. Later texts, e.g., *Artis cuiuslibet consummatio*, that are both comprehensive and practical repeat PG's methods. Planimetry ends shortly, and turns to the more exacting Cosmimetry.

Cosmimetry (#39-57), it seems, is addressed not to practitioners, but to schoolmen. It is the least adequate part of PG. Hugh explains Eratosthenes' way to compute the earth's circumference, which was found to be 252,000 stades (see Appendix D). The earth's diameter, therefore, was 80,181+1/2+7/22 stades, a value somewhat larger than that given by Macrobius. The labored reading of the translation, especially in #40-48, reflects the Latin text's clumsy, and inconsistent, use of Latin number words and Roman number symbols (replaced here by Arabic notation). PG does not (and cannot) give an absolute length for a stade, though *De arca Noe morali*, and Gerbert, provided a more detailed list of measurement units and their relations (see Appendix F).

PG explains how unidentified "Egyptians" computed the height of the sun (*altitudo solis*). The sun is a point source of light whose rays strike the flat earth not in parallel lines, but at different angles. The argument uses similarity theory of right triangles. Hugh did not observe that this replaced the arc of earth's surface with the level line base of a right triangles, but he thought it worked in Egypt because of "the flatness of the land and the proximity of the sun." The sun's height is 4,820,000 stades, the solar orbit diameter ("*integra solaris circuli diametron*") 9,720,181+1/2+7/22 stades. Macrobius's estimate (9,600,000 stades) is summarily dismissed.

But both PG and Macrobius are in immense error by today's standards.

Though methods of "*prisca sagacitas*" work in the flat Egyptian desert where the sun is directly overhead, Hugh feels constrained to propose ways usable elsewhere (#42). To use a "two-station" method of altimetry supposes a flat earth and non-parallel solar rays. Two observers take simultaneous noon time readings of solar declination. Success demands separation of the two along an accurately measurable meridian line long enough to allow discernibly different quadrant readings of the declination angle. The separation would be something of the order of 70 miles (about one degree of latitude), an unlikely situation in western Europe for this method. Did Hugh try it? Could it work with the measuring devices then available? PG gives no experimental data to confirm the validity of the process.

Another, supposedly easier, method uses circle geometry (#43). Details in PG are incomplete, but if the reconstruction proposed in Appendix E is correct, the method has a fatal flaw. Here, too, PG provides no evidence that anyone has in this way gotten results consistent with earlier values for the solar altitude.

PG next computes the solar diameter (#44). Since there is (incorrect) data for the length of the solar orbit, then, if the solar diameter is computed as a known fraction of this length, the diameter itself is readily known. There are two ways to compute this fraction. The first, taken from Macrobius's *Commentary on the Dream of Scipio* (I, 20,25), is unworkable and theoretically invalid. At equinoctial sunrise a gnomon is used to measure the time for the solar orb to rise completely above the horizon. The positions of the two sun shadows, one at first visibility, the other at full visibility of the orb, are compared, and, Macrobius says, are found to make one ninth of an hour in the equinoctial day, i.e., six and two-thirds minutes. A solar diameter is then one 216th of the solar orbit, or

$$141{,}431 + 1/6 + 1/24 + ((1/216)(1 + 5/6 + 1/7(6)))$$

stades. Among ancient estimates of the sun's apparent size, that of Macrobius, at one degree, forty minutes (a 216th of the circumference), is the largest, more than four times as large as Ptolemy's mean estimate of 33'20", itself close to the actual mean apparent size of 31'59" (Stahl, 1952, p. 253).

The time needed to see the sun rise above the horizon depends on where the observer is. Sunrise is slower on the Arctic Circle than at the Equator, for the tangential velocity of a point on earth's surface about the polar axis depends on the latitude of the point. An

observer in Scotland will record a longer sunrise time than someone in North Africa. Their data will differ, and so too the values they compute for the sun's diameter. Did Hugh realize this? He mentions unidentified objections and, to allay them, gives another method, with astrolabe rather than gnomon, to get not the elapsed sunrise time, but the angle subtended by the newly risen sun. This can be done. If the first method gave correct results, this should corroborate them, and the experimental data should show that the rising sun subtends (1/216)360=1.66 degrees. But PG gives no data gotten by the alternate method and does not consider the question of the consistency of the two methods.

PG can now compute the length of earth's shadow (#45), an easy problem that the Greek astronomers studied to account for lunar

eclipses, though PG does not mention eclipses, compute lunar or planetary distances, or seem aware of the astronomy of Aristarchus and Apollonius. The text, unfortunately, has unsatisfactory manuscript witnesses (noted in Baron's critical apparatus, especially for #47), giving computational results that cannot be explained by the numerical data in the text. The process is correct; incorrect results are drawn from (incorrect) data. Thinking the solar diameter less than double the earth's diameter, PG concludes that the earth's shadow must stretch beyond the antipodal point of the sun's orbit by 2873+1/3 stades. Macrobius said that it reached only to the solar orbit; PG corrects him. The text of #48 refers to a diagram illustrative of the argument that is not found in any known PG manuscript; one has been supplied here.

Six short sections (#51-57) give obscurely worded statements about geometric optics, preparatory to a geometry of the visual horizon. Interpretative figures have been supplied for them. A final wry comment about Macrobius, a promise of more lectures, and, only in manuscript M, the epigram: "All passes away, save love for God; most opportune, then, this enduring love," close *Practica geometriae*.

Some general observations and a question make an appropriate close of this review. Evidently here in PG, as in *Didascalicon*, Hugh honored his predecessors, explaining their accomplishments as he knew them, in disciplined and logical sequence. But did his admiration for their work inhibit a critical sense, or divert attention from the need, at some time, to justify PG's basic and recurrent methods? We have noted that Hugh seemed to recognize the need for alternate methods to compute solar altitude and solar diameter,

though he did not specify the objections raised against the work of the "Egyptians," or compare their results with those of alternate methods.

Hugh reverenced human intelligence and knew that its power was not exhausted with these accomplishments. This altimetry section, most of all, shows consistent use of basic principles. Cosmimetry, by contrast, does not ask and perhaps cannot answer whether its methods are well founded in theory. Though directions are to be followed with care, and measurements done "with great care (*cautissime*)," the reader is not urged to compare results or repeat experiments. Yet PG repeats itself often, and on occasion the reader is left to fill in details, almost as a student exercise. Cosmimetry is very much a scholastic, not a practical treatise.

PG's resources were chiefly a section of the agrimensorial legacy that did not know abstract notation in geometrical matters. Gerbertian texts show Arabic numerals, but none are to be found in PG's text, which used

number words and number symbols indiscriminately. PG, in short, provided practitioners and schoolmen with vocabulary, problems, a sense of structure, an agrimensorial legacy, and an inchoate spirit of inquiry. It was still a time for assimilation of the simple, and profound, ideas that would flower shortly when Euclid, Archimedes, and Apollonius were heard once again in the schools of Europe.

Prologue

My goal is to teach practical geometry to our students, not as something new, but rather as a collation of older, scattered material. Say what you will, I think our predecessors worked miracles. They had immense energy, and tried to get at the truth. Hard work could not dampen their ardor, nor any obstacle deter their efforts. They had deep insight into marvelous and almost incredible matters, and even in lesser ones they provided many examples of wisdom. To equal them may not be possible; not to try would be a disgrace. But enough exhortation; let us address our task.

Prenotanda

(1) Dimension has three modes: length, width, and height. Geometers call length a line. Width is a surface, and height, in technical terms, a solid. Scientific geometry deals with all three. A line is an extension from any one point to any other, and in any direction: forward, backward, right or left, up or down; as long as there is extension, nothing else is needed to satisfy the nature and definition of a line. A point has the power to let a line emerge from it in any direction, and to come to it from any direction. A single point stands in the middle of every direction, because it becomes the center of a circle when an outer circumference intersects lines from it in every direction. For lines from the point have power to cut the circumference at each point, and every circumference has the power to send equal lines to its center from every direction.

(2) A surface occurs if one or more lines are adjoined to a given line on its flank to mark out a width. A solid occurs if one or more surfaces are put atop a given surface to build a height. Its length is

between end points, its breadth within boundary lines, and its depth between surfaces. Length is measured between endpoints, breadth between end points of lines, and depth between end points of surfaces.

This said, we note that geometry is either theoretical (speculative) or practical (active). Theoretical geometry uses sheer intellectual reflection to study spaces and intervals of rational dimensions. But practical

geometry uses instruments, and gets its results by working proportionally from one figure to another.

There are three kinds of practical geometry: altimetry, planimetry, and cosmimetry. In each, practical geometry calculates line length; in altimetry, extension up and down, in planimetry, extension forward and backward, right and left, and in cosmimetry, circumference length. Altimetry is so named because it studies what is high or deep. Note that sometimes terms vary: the lofty is called deep, and, vice versa, the profound, lofty. We speak of the high seas, and the depth of heaven. All quite proper, because what descends from above, or arises from below, has extension. One and the same length is high, or, seen the other way, deep. Planimetry occurs for extension in the plane. Cosmimetry gets its name from the cosmos. Cosmos, in Greek, means world, and cosmimetry is measurement of the world. It measures the circumference we consider in the ambit of the celestial sphere, and other celestial circles, as well as earth's circumference, and many natural circular phenomena.

Weighty topics, indeed, and remarkable ideas, well worth study. To the novice, geometry seems to offer knowledge beyond belief; to the expert, all is familiar. Here we plan to explain practical geometry under the three headings: altimetry, planimetry, and cosmimetry. We will indicate what instruments to use for a given measurement problem, and how to get numerical data with those designed for the purpose.

First, to introduce in a coherent way the nature and shape of the objects we study later, we consider a simple figure, namely, the right triangle. It has three sides: base, perpendicular, and

hypotenuse. From their mutual relations, you may at once compute what has to be measured, whether level, vertical, or orbital, provided you have mastered the ideas here.

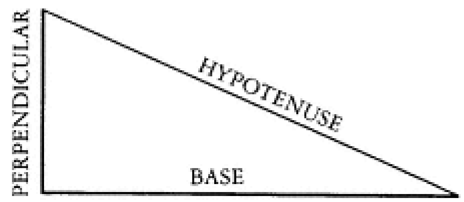


Figure 1

(3) Consider a right triangle. Its base is the lower line, set along the ground. Its perpendicular is a line at the base end, raised straight up, so that it inclines to neither side, but makes a right angle, where it meets the base. The hypotenuse is a line from the top of the perpendicular to the far and opposite end of the ground line (i.e., the base). It descends obliquely, and so makes an acute angle at both ends. (Fig. 1)

This triangle can never have all its sides equal. Sometimes there are no equal sides; sometimes two, i.e., base and perpendicular. The hypotenuse never equals the other two. A rectangle, however, drawn to a base set to a perpendicular, is always double the triangle within base, perpendicular, and hypotenuse.

There is much more about right triangles, but it must be said elsewhere.

We now examine how the right triangle scheme can be found in geometric configurations. One must fit the shape of the world to this device. Just as large bodies reflected in mirrors seem small, and the small images keep the proper proportions of the large bodies they represent, so here the vast spaces our minds cannot comprehend, must be reduced, by rational procedures, to model form, to be scientifically tractable.

The entire world sphere surrounds earth, set as a point in its center. It is equidistant in every direction, and convex in curvature everywhere. To us, on the surface of the earth, it appears stretched out and joined afar in every direction to earth itself. This, of course, is not the case, but appears so, because straight sight lines cannot differentiate great distance.

Our sight line terminates, in every direction, in a circle we call the horizon. Here, as noted, the celestial sphere itself seems joined with earth's surface. Horizon is "limit," because it limits the extent of our vision, and somehow prevents it from going farther.

The horizon, then, is a circle bounding that portion of earth's surface open to our view. The celestial hemisphere sits over this circle. If we draw straight lines from its middle (i.e., center) into the four quarters out to its circumference, we divide the horizon into four parts.

If we use the hemispherical convexity to draw circumferences from the ends of both diameters, through the middle, to antipodal points, we split both upper and lower hemispheres into corresponding quadrants. Then we find, in each quadrant, a right triangle with perpendicular rising from the center of the horizon circle to the upper hemisphere pole. If the

line through the center drops to the corresponding lower pole, we find other similar triangles formed there the same way. (Fig. 2)

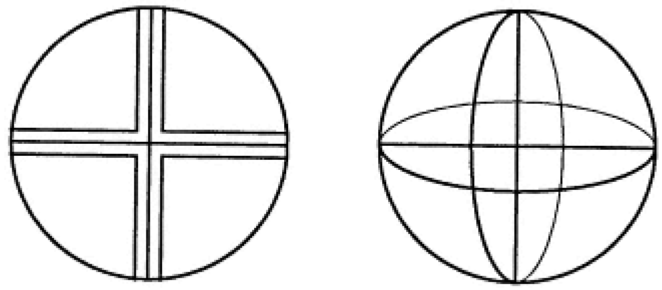


Figure 2

(4) From east to pole there will be a triangle whose perpendicular is a line erected from the horizon center to hemisphere pole. Its base is a line stretched from center to circumference along the ground (earth's surface). Its hypotenuse, however, is a line taken from the point where the base intersects the circumference, i.e., the horizon, along the hemisphere surface, to the top of the perpendicular dropped from the pole. (Fig. 3)

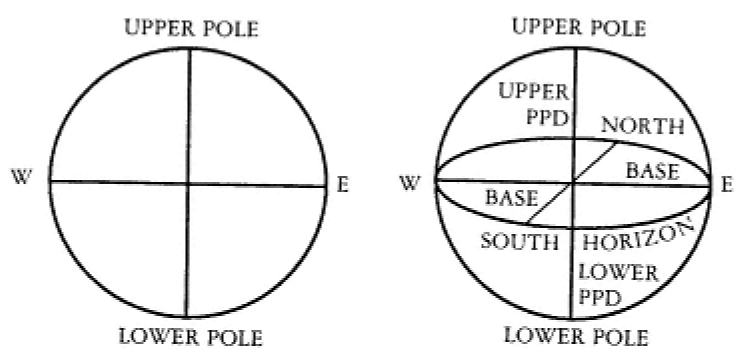


Figure 3

In the same way, there is another triangle from the pole to the west, a third between the west and the lower pole, a fourth between lower pole and the east. A like case starts from the north. There, in the same way, we get a triangle between horizon and upper pole; next to it, another, from pole to southern horizon. A third triangle

reaches from the southern horizon to the lower pole, and a fourth from there to the northern horizon.

(5) And so we structure eight triangles over the two sides of the celestial sphere. All have the same form and measure; we distinguish them by horizon and pole. They share the same base in pairs; four are symmetrically set around a common perpendicular. Each has a unique hypotenuse.

To synopsize: There are three circles. One, by position, divides the celestial sphere into top and bottom. It goes around the rim of the earth, from east, to north, to west, to south, and back to east. Another circle goes from east, through the upper pole, to the west, and from there, through the lower pole, back to east. The third starts at the north, goes through the upper pole to the south, and from there, through the lower pole, back to north.

Three straight lines, drawn as diameters from circumference, through the center, to circumference (one from east to west, a second from north to south, the third from upper to lower pole), divide each circle into four equal quadrants. Each quadrant has 90 of the 360 degrees of the total firmament ambit.

The four upper triangles are drawn about the same perpendicular, as are the four lower ones. They have a common base in pairs; those from the east to upper and lower poles, and similarly, those from west, north, and south, to upper and lower poles. Drawn from the four corners to the pole, the hypotenuse is named the *anabibazon*. The term means "that which arises."

But remember this about the right triangles. If, of their three lines, the two about the right angle are tilted toward each other, this makes a pair of opposite acute angle triangles, and a pair of opposite obtuse angle triangles. (Fig. 4)

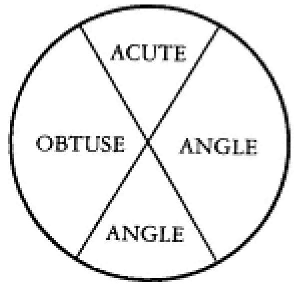


Figure 4

(6) In addition, a perpendicular drawn orthogonally within any right triangle will have the same ratio to the base and hypotenuse it intercepts that the larger perpendicular of the triangle has to its own base and hypotenuse. (Fig. 5)

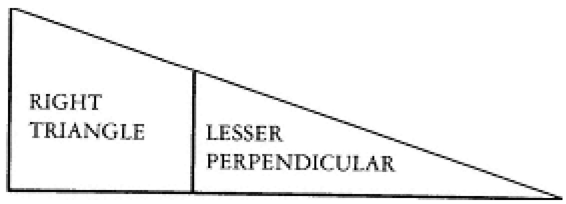


Figure 5

Suppose, next, a right triangle attached to the hypotenuse somewhere, but drawn with another base. It has the same triangle ratio set, because all right triangles under a common hypotenuse are, for this reason, necessarily similar. (Fig. 6)

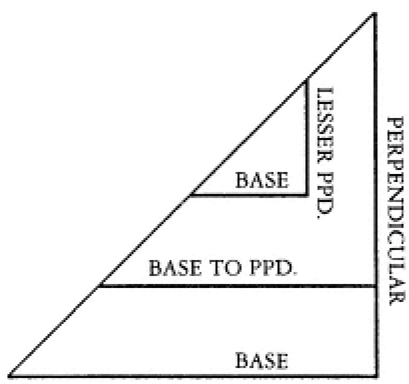


Figure 6

We have treated these topics in detail to stock the reader's mind with basic principles for easier understanding of what follows. Let us go to our first topic, altimetry.

I. Altimetry

(7) Any high object we see generates a triangular form. Its perpendicular is the height, and its base the ground line from the foot of the perpendicular to the spot where the height can be seen with unaided eye. For from the spot where sight first perceives the object rising in the distance, a triangle forms. Its perpendicular, as noted, is the tall object, its base the plane, its hypotenuse the sight line extended from base end to the top of the height. Triangle base and hypotenuse both end where the sight line focused on the object, first meets the eye at ground level.

(8) Measurement of a height between ground and vertex, or vice versa, is expressed as elevation or declination, measurement of a depth between ground and nadir, or vice versa, as declination or elevation. Circumference measure is always taken along the *anabibazon*, which, in the four quarters, rises from horizon to pole. Planar extent is computed along a base, in any direction, from horizon to foot of perpendicular.

The visual triangle, however, that has all these dimension ratios, gets steeper as the perpendicular gets bigger by approach to the vertex line, until base and hypotenuse are lost, as the perpendicular reaches the vertex line. But as the perpendicular, alone, decreases by recession to the horizon, the triangle falls off, until only the base remains, and the other two sides disappear.

Just as the triangular form, in the first case, rising from its base, collapses into the perpendicular, so, in the second, starting from the vertex line, it falls to base only. First, perpendicular only increases, until base and hypotenuse vanish; then, perpendicular decreases

until only base remains, as perpendicular and hypotenuse vanish together. The actual height, however, naturally stays the same, as high when first perceived, as afterwards, when, moving away, it drops from sight, a link between either end.

Note that from the horizon, along the *anabibazon* ascent, to the pole overhead, is a quarter circle, and that from the horizon to the circle center is half the diameter. Now the ratio of quarter circle to semidiameter is one and four sevenths. If I draw a straight line hypotenuse between perpendicular and base (circle geometry has them equal because both reach from center to circumference), I can prove the hypotenuse less than the quarter circle. (Fig. 7)

(9) Suppose I raise another perpendicular on the far side of the triangle base. Then, if the diagonals intersect each other at midpoint, the

perpendiculars must be equal. So two opposite and equal triangles are set up on the same base. (Fig. 8)

But if the diagonals intersect beneath the middle, this shows that the perpendicular on the far side is the smaller. If they intersect above, it is the greater. And so a ratio between the two perpendiculars is established according to diagonal intersection. (Figs. 9-10).

If you understand the right triangle, you can readily understand our measurement rules.

(10) The astrolabe is surely the surveyor's most important instrument. We must explain it before we set out measurement rules. On its postern face, there is an equilateral rectangle, indispensable for geometric measurements, set up beneath the median line between occident and lower pole. Here is a description.

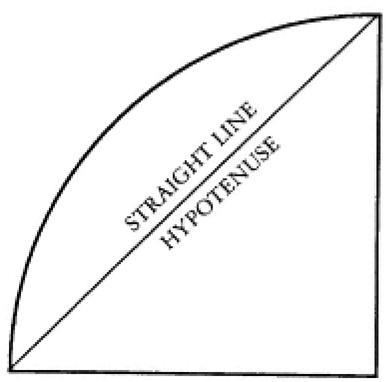


Figure 7

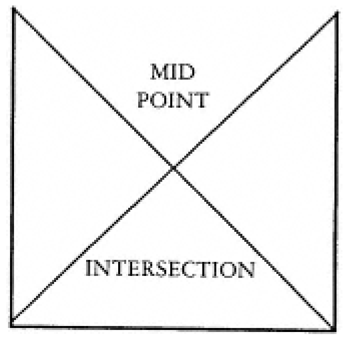


Figure 8

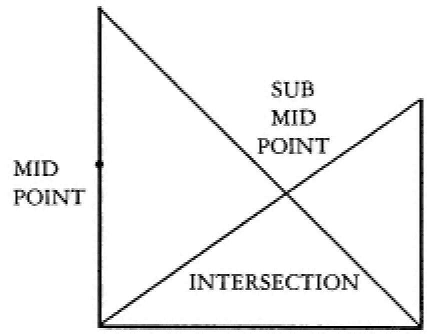


Figure 9

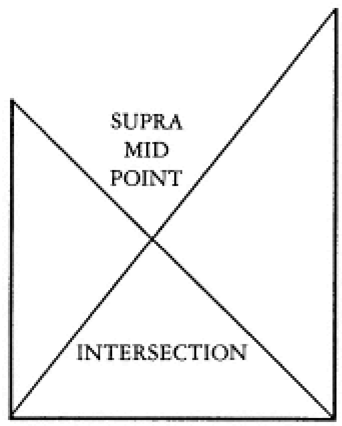


Figure 10

First, consider three quadrants of the whole circle, i.e., the underside one between the occident and lower pole (the rectangle is to be set in it), and two more in both parts of the other side, one quadrant curving forward from the left, the other backward from the right. Each quadrant is divided into two equal parts, and midpoints marked. Straight lines are drawn to these points from the center. (Fig. 11)

So, at the middle is an equilateral rectangle. Two of its sides meet in the center of the circle, and the other two form an angle about the diagonal in the middle of the quadrant arc. Two more angles are formed, above and below, on the opposite side about two diameters

of the circle. Divide each of the two rectangle sides making an angle in the quadrant

arc into twelve parts. Pair them systematically into intervals within; six altogether for each side.

(11) Suppose I direct the astrolabe to a prominent height, and point the alidade to its apex. Two equal triangles are formed on opposite sides; one under the median line (which acts as horizon), the other above and opposite. In both, the medicline (alidade) direction fixes the hypotenuse, though in opposite settings.

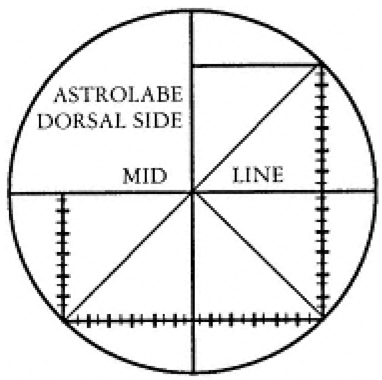


Figure 11

In the upper triangle, the perpendicular is upright, as in nature. Its base underneath forms an angle with the hypotenuse at the center of the circle. In the other triangle, however, formed opposite under the median fine, the natural order, somehow inverted, appears reversed, like an object's shape reflected in water or a mirror. Here the perpendicular falls from the median line (which also acts as base), so that the hypotenuse, which intersected the first perpendicular at its apex, and is extended in a straight line, intersects it as far beneath the median fine, as it did the first above it.

So there are three triangles. Two, diametrically opposite within the instrument, are equal. The third, outside, has the object height as

perpendicular, and, as base, the ground line from the foot of the object to the surveyor's station. Its hypotenuse is the sight line from the surveyor's eye to the object's apex.

Although this triangle evidently differs very much from the other two in size, it shares their recognizable common proportions. By our rule, if a perpendicular is set up at right angles inside a right triangle, it will have

the same ratio to whatever part of base or hypotenuse it cuts off, as the larger perpendicular has to its base or hypotenuse. This proves that in the upper quadrant triangle, the perpendicular has the same ratio to its base as the distant height has to its. The line from surveyor's eye to foot of the height is the larger triangle base. Suppose a perpendicular raised from some spot on the base up to the sight line for the apex of the object. Then the base has the same ratio to the perpendicular (i.e., the height), as its part, cut off by the smaller perpendicular at its foot for a base, has to the latter.

The idea needs development since the upper triangle perpendicular does not have as base all the larger base cut off at its foot, but only half (the other half is left to the opposite and lower triangle). Suppose the object far away. Point the astrolabe in its direction. From its median line, draw a base line to the foot of the object on the horizon. Bring a hypotenuse for the triangle down from the object's apex to the surveyor's eye. (Fig. 12)

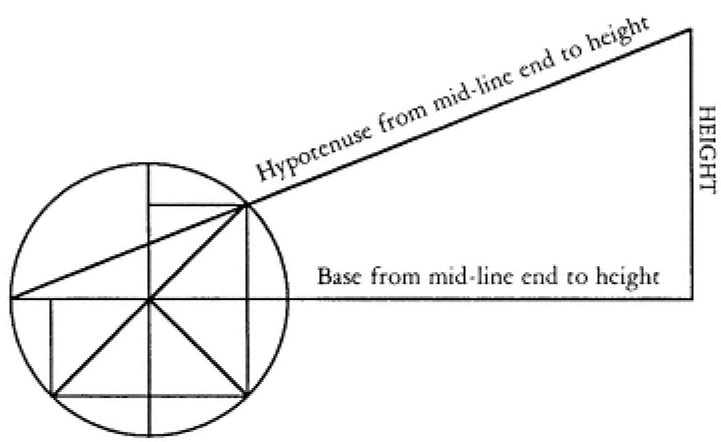


Figure 12

(12) Do this first, disregarding the medicline, so that we can see how a perpendicular should be set up, for that base part equal to the

instrument's length, to generate the larger triangle's proportions. Clearly, should the medicline have its axis where hypotenuse intersects base, and then, along the hypotenuse extension on the other side, be directed to the

apex of the object, the perpendicular to the medicline vertex would both be larger, and have a larger base inside for the resultant triangle. But here, because medicline axis is at diameter center (so that when raised on one side, the medicline is equally depressed on the other), the ratio of inner to exterior triangle base is that of upper perpendicular to object height. (Fig. 13)

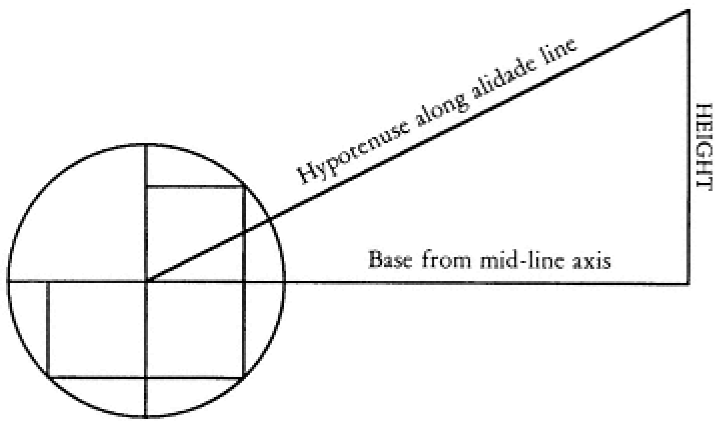


Figure 13

(13) As a result, the medicline, with central axis, generates a similar triangle. Of course, one would also be formed were the medicline elevated, at the common end of great base and hypotenuse, to the apex of the object, but had its perpendicular on the other side of where it is now.

So we have proved all these triangles similar, and may conclude that the ratio of base to perpendicular in either medicline triangle is certainly the ratio of distance between surveyor and object to the object's height.

This is true if the object is on the far horizon. But if it is nearby, the median line does not lie in the larger triangle base. Rather, the instrument triangle, though similar, is attached under the larger

triangle hypotenuse, and grounded on a different base. The surveyor's height must be added, either in proper proportion to the rear of the base, or just as it is to the height to be measured, to preserve triangular ratio along the extent of the larger triangle hypotenuse. (Fig. 14)

(14) Keep in mind that, in reality, the base changes and the perpendicular is fixed as the object approaches or recedes. But in either rectangle in the astrolabe, first the perpendicular is variable and the base fixed, then, with order reversed but ratio the same, the base is variable, and the perpendicular fixed. The medicline marker reading, on the right side, is compared to the total rectangle side, so that, while the rectangle side that acts as base stays numerically fixed, the perpendicular reading varies as the surveyor approaches or recedes.

When the medicline stands at rectangle corner, height equals base. Beyond it, roles are reversed, with the perpendicular exceeding its base. When the medicline reading is compared to the entire rectangle side, the base is seen to stand in this ratio to perpendicular.

Now that we know the nature of the instrument, we can give measurement rules. First, we explain how to measure any height on a level surface.

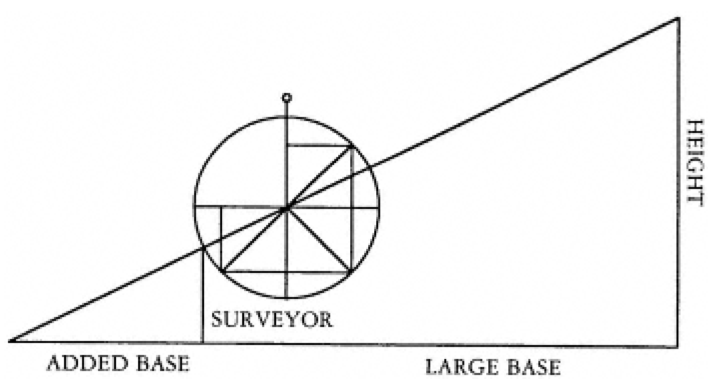


Figure 14

(15) To measure the height of a tall object set on a plain, the surveyor takes an astrolabe and sets the medicline at the rectangle

cornerpoint. Then, with the astrolabe directed to the object he wants to measure, he moves back and forth until he can see the top of the object through both medicline apertures. When he can see it, he knows its height is equal to the space between himself and the object plus his own height.

(16) Having said he must add his height to the base, we should explain what this height is, and how to add it. The surveyor's height is the distance from the ground to his eye whence comes his vision line. For it, surveyors often have a rod of this length to expedite the process.

The surveyor's height is not always added the same way. The rule is that as much length must be added to the base of the greater triangle, behind the surveyor, as the hypotenuse, drawn from apex to surveyor's eye, would intercept on the ground, were it extended behind him. With this, another triangle would be formed behind him whose perpendicular would be his height, and its base the addition. Because this triangle must be similar to the others, the amount added must be computed according to the previous astrolabe ratio. That is, the addition must have the same ratio to the surveyor's height as the base of the astrolabe triangle has to its perpendicular.

(17) To measure the height of an object in front of you without moving from your place, raise the astrolabe to the object. Adjust the medicline until you can see the top of the object through both apertures. Then compare the medicline degree reading with the whole side of the square. The ratio of the medicline reading to the whole side (i.e., to twelve) is the ratio of height to intervening space with surveyor's height added either proportionally or exactly.

(18) If the intervening space is impassable because of an obstacle such as a river or a gorge, you can still get your result. Use the astrolabe where you are. Adjust the medicline to the top of the object until you can see it through both apertures. After this, note how many degrees of the side of the square below appear above the medicline. Compare them to the 12 degrees of the whole side. By rule, this is the ratio of height to intervening distance plus surveyor's height.

Next, move back some distance to a second position. Take the astrolabe, and sight the top along the medicline. Record the

medicline degrees on the square side and compare them to the whole side. The ratio is now that between height and intervening space plus surveyor's height. Then compare the first and the second base, to determine how much the second exceeds the first. Now compute the length of the first base by means of the difference between first and second, i.e., the distance between your first and second positions.

For example, suppose the medicline marker reading at the first station is four. Because twelve is the triple of four, the intervening space plus surveyor's height will be triple the object's height. Suppose the medicline marker reading at the second station is three. Because twelve is the quadruple of three, the space plus surveyor's height will be quadruple the object's height.

So suppose the first station separation plus surveyor's height (the first base) is triple the object height (the perpendicular), and the second station separation plus surveyor's height (the second base) quadruple the object height. Clearly, the second base is one and a third times the first. A third part of the first base will be the excess of the second over the first. Evaluate this distance, and take it as just one third of the first base.

A warning. This distance is not always that from first to second station. The surveyor's height adjustment is not the same at both. Rather the distance is measured from the end point of the first addition (where first base ends) to the end of the second addition (where the other ends). This gives the true difference between bases.

(19) Another measure rule for inaccessible heights differs from this one only in details. Divide the marker reading noted at the first station into the whole square, i.e., into 144. Do the same for the second station reading. Take one part of the first division, and one part of the second, and compare them. Subtract the smaller from the larger. Compare the remainder with the entire side of the square, i.e., 12, to compute their ratio. This is the ratio of the object height to the intervening space between the two stations, each adjusted by the surveyor's height.

For example: the first station medicline marker reading is four. The total square, i.e., 144, divided by this is a quarter part, 36. The second station marker reading is three. The total square divided by three is a third, 48. Of the two 36 is the smaller; subtracted from the larger 48, it leaves a remainder of 12. Comparison shows this

equal to the side of the square. Therefore, the height equals the distance between stations.

But remember: whenever the medicline rests on the vertical side, the variable reading is the perpendicular, and the base is fixed. When the medicline is on the other, the variable reading is the base, and the fixed one the perpendicular.

(20) Another rule for height uses the shadow cast by an object on level ground. On a sunny day the surveyor takes his astrolabe, and directing the medicline to the sun beam, adjusts it so that the solar ray shines through both medicline apertures. He then compares the medicline marker reading noted on the square to the whole square. The ratio of object to its shadow is exactly that of medicline marker reading to the side of the square.

(21) Many other devices can measure the height of an object on a plain. The right triangle (called "Pythagorean" after its discoverer) is perhaps

the most important. Take one whose base and perpendicular are in sesquitertial ratio (it makes no difference whether base or perpendicular is larger, as long as the ratio stays fixed).

The device is used like this. Lining up the perpendicular with the height, the surveyor moves tentatively until the apex of the unknown height appears across the top of the triangle perpendicular, in line with his eye, at ground level, at the triangle base end. When it appears, he knows that the intervening space from his observation point to the foot of the height is in sesquitertial ratio to the height itself, for the triangles are similar.

(22) Another right triangle has base and perpendicular equal. The surveyor locates it in a like spot, so that the top of the height can be seen from the end of the triangle base across the top of its perpendicular. By the law of similar isosceles figures, the intervening space is found equal to the height.

Some practitioners attach stands under their triangles, or even hold them manually, since many find it injurious to bend down often and get their eye on the ground. Then, of course, the surveyor's height must be adjoined proportionally to the large base, away from the object. Whatever the proportion of the triangle, the ratio between intermediate distance and unknown height will be the ratio of triangle base to perpendicular.

(23) Others put a rod, double their own height, between themselves and the unknown height. At its midpoint, they attach, at right angles, another rod equal to their own height. The second rod, half that to which it is attached at right angles, acts as a base, dividing the first one at its midpoint. They move this composite device

backward and forward until they can see the top of the object lined up with the ends of both rods. When this happens, they know the intervening distance (with surveyor's height added) equals the height.

(24) Still others employ a rod of arbitrary length, but taller than themselves. Either they keep their place fixed and position the rod in an appropriate spot, or they fix the rod and move until they can see the top of the object across the tip of the rod.

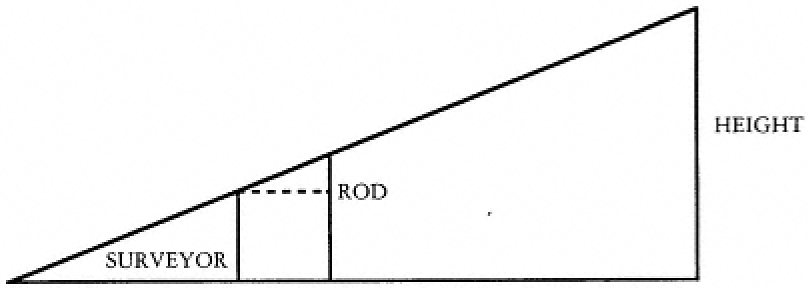


Figure 15

Next, they compare the space between surveyor and vertical rod to the part of the rod above the surveyor's eye level. Evidently, this is the ratio between the distance from surveyor's foot to the object's, plus surveyor's height, and the height itself. (Fig. 15)

If the intervening space is impassable, a second position to the rear is needed, just as with an astrolabe. The vertical rod may be moved, or not. Suppose that, at first position, the sight line goes across the top of the vertical rod to the tip of the object. Then a duplicate vertical rod must be set up in a second position. But if the sight line went beneath the rod's top, the rod may stay in the same place for a second sighting.

(25) Others, to measure a height, in appropriate circumstances, choose a rod of one or two cubits, more or less, and with it measure as much distance from the foot of the object as needed for the task. Within this space, they fix a rod into the ground so as to cut off from the space, at its far end, the length of the rod. Next, they lay supine on the ground, facing the height in a spot such that their sight line is directed from the end of the measured distance to the top of the object. If the sight line goes from viewer on the ground, across the top of the intervening rod, to the top of the height, the measured distance is the height. If the rod is too short for the

object's height, the height is more than the measured distance; if the rod is too large, it is less.

There are many other ways to do this. Less skillful surveyors, with no equipment or training for better methods, judge a height by shadows. They take the ratio of a stake set in the ground to its shadow as that of the unknown height to its shadow.

(26) Some divide the object's shadow into equal parts. Then they fix in the ground a short rod of length equal to one part, and compare the shadow cast to the rod itself. If shadow exceeds rod, they subtract this excess from each part of the larger shadow. What remains of the larger shadow is the object's height. But if rod exceeds shadow, they add the excess amount to each part to get the object's height.

(27) Some practitioners use a bowl or basin filled with water, and put between themselves and the object, to estimate a height. The basin center is dearly marked. Then they move away from it until they see the top of the object reflected at the center of the basin they set. Next, they measure with care the distance between observer and the center point of reflection. The ratio of this distance to the observer's height is that of the far side distance to the object's foot to the height itself.

(28) Others use a mirror put in the middle of a field, or held in their outstretched arms. They can calculate the height the same way, noting its reflection in the polished surface before them and the center's place. Note that if the mirror is above ground in the observation process then, for comparison purposes, you must use only the height of the object and the surveyor's height above mirror level.

Enough now for this problem. Our next task is to measure a height either in a valley below, or on a hill above us. More difficult, perhaps, but reason illuminates Nature, and everything is open to understanding.

(29) Anything rising from earth with vertex raised on high,

towering over flat ground, is a height. Suppose we have to measure it from a spot on another level, e.g., into a valley from a hill, or onto a hill from a valley. The surveyor should first fix the natural horizon for his station. The median line determines this everywhere without deviation. Once this is found, he must consider where the unknown height is set, to determine how far it is under his horizon (if below), or how far over it (if above). After he has added the height of the object, he should take the total as perpendicular. When he has determined the ratio of the complete triangle, he must subtract the height of the hill. The remainder is his answer. This is the procedure for objects on higher levels. For lower level heights, there will be, on one side, the surveyor's height plus the hill's height as perpendicular, and the triangle must be set up with this.

To clarify. Suppose the height on a hill, the surveyor in a valley. Let him turn the astrolabe to the hill and unknown height. With medicline set on the median line, let him note carefully through both apertures the spot opposite on the hillside corresponding to the median line. The

surveyor should then turn the medicline up the hill side to the top of the object, and take as perpendicular the whole length above the horizon (upper hill height plus object height). The ratio of this composite length to the length of the intervening space is that of the medicline marker reading on the lower square side to the whole side. Next, he should depress the medicline to the foot of the object (the top of the hill), and compare medicline reading with square side, as before. He must determine what ratio the hill's height has by itself to the same base, for the difference between the first and second perpendiculars is precisely the height of the object. But since the base (intervening space), from surveyor's eye to foot of the perpendicular, is impassable because of the bulk of the hill, readings must be taken at a second station to the rear, as already explained.

(30) Suppose the tall object set below us. Direct the astrolabe toward it, and depress the medicline beneath the median, with the square turned in the same direction, until the sight line runs through both apertures to the foot of the object. A triangle is formed whose perpendicular is the surveyor's height plus the height of the mountain above the plane of the object. Its base is the line from the center of the mountain (the foot of the perpendicular) across the plain to the foot of the object. With medicline set, we get that the ratio between perpendicular and base is that of medicline marker reading to square side.

Because the intervening space is impassable, a second station is chosen to the rear along the plain. Once again the sight line is directed through the medicline to the foot of the object, and the medicline degree reading compared to the square side, to get the

ratio of base and perpendicular. Once this is found, one can easily determine the lengths of the entire first and second bases through the interval between first and second stations, as we have already explained. This done, the surveyor, at his second station (or at the first, if he so prefers), raises his astrolabe to the object, and adjusting the medicline not to its foot but to its apex, sights the top through the two apertures. The medicline reading is noted and compared to the square side, and so he learns the ratio of the object's perpendicular to the base.

These results are now collated. The surveyor realizes that three triangles have been formed in his three observations. In each, the perpendicular is the same, but the base is always different. The first triangle, formed at the first station, has as perpendicular the height of the mountain plus the height of the surveyor. Its base stretches as a line from the lowest point of the perpendicular (at the mountain floor) over to the foot of the object set on the plain where the sight line drops to the

ground. The second triangle, formed at the second station, has a perpendicular equal to the previous one, but its base is longer only by the distance between first and second station. But the third triangle (also formed at the first station), has the same perpendicular as the first two, though a different base: one, in fact, that stretches from the foot of the first perpendicular to the place where the sight line, crossing the top of the object, descends along the hypotenuse extension (on the far side of the object on the plain) to meet the ground.

In this scheme, the first and second bases have the same end point, but not the same initial one; first and third bases start from the same perpendicular, but do not end in the same place. The length of the first base is found with help of the second, and by means of the first base the length of the first or second perpendicular is determined. And now one can readily determine the length of the third base by comparison with the first.

In this third triangle, then, one can surely determine with this data the length of either base or perpendicular. Within this triangle is described a triangle whose perpendicular is the height mentioned above. Its hypotenuse is the sight line going across the top of this height and carried to the far side. Its base is the far side space the hypotenuse intercepts. This triangle is necessarily similar to the larger triangle. Its base is a part of the larger one, and if it is known, then its perpendicular is also.

(31) There is an alternate theorem for measuring an object from a higher elevation. It is quicker, provided the height of the mountain the surveyor stands on can be measured either down a sheer cliff or by some other way.

Suppose the valley below, where the height to be measured is set, uniformly level. Suppose at the far end a sheer mountain or cliff, whose height above the level valley floor is known. The surveyor should stand at the edge of the cliff, looking down at the object. Then, moving back a bit, he sets up in front of himself a rod, perpendicular to the ground. The rod should be smaller than his height, or, if he wishes, equal. It should be situated so that the viewer's sight line going down from its tip, across the edge of the cliff, ends at the foot of the object. (Fig. 16)

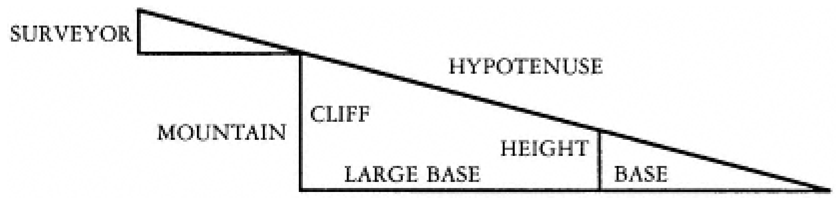


Figure 16

With this, there are two similar triangles, a larger and a smaller. The smaller perpendicular is the rod in front of the surveyor; its base the distance from the foot of the upright rod to the edge of the sheer cliff. The larger perpendicular is the height of the cliff; its base the line below reaching over to the foot of the object. Both triangles are set orthogonally under the same hypotenuse, and so are necessarily similar. Thus the larger perpendicular has the same ratio to its base as the smaller to its. The length of the smaller perpendicular, its base, and their ratio are all known, as are the larger perpendicular length and its ratio to the base. So the base length is known. (If there are two segments, and the length of one is known along with its ratio to the second, the second length is also known.)

Next, the surveyor, with eye at midpoint of the rod (in the same spot as before), judges how far down the sight line descends from cliff edge to the top of the object. Projected beyond this, the sight line should now form three triangles. The first has as perpendicular the section of the rod down from the surveyor's eye, and its base is the same as before. The second has the same perpendicular as before (in the second triangle), but its base is different. The perpendicular is, indeed, the cliff height, but its base the line from the foot of the perpendicular, now, not to the foot of the object, as before, but beyond, out to where the sight line, coming across the top of the object, meets the ground. The third triangle, inside the

second, has the object height as perpendicular, and as base, that part of the large base between the object and the far intersection of hypotenuse with base. In the first triangle, lengths of perpendicular and base and their ratio are known; in the second, length of perpendicular and ratio to base are known, so length of base is also. In the third, length of base is known (by length of the second base), and its ratio to perpendicular. So perpendicular length, that is, the height, is known.

(32) Depth measurement, between horizon and lower vertex, is done through the quadrant and the quadrant opposite the upper one, either in elevation or depression. First, any depth, to the extent it is seen, can be measured with an astrolabe like this. The mensor judges the well's diameter length, making sure its cross section is circular. This done, he stands at the well edge, and turns the medicline until he can see, through both apertures, the end of the wall opposite. Then he relates the medicline numerical degree reading to the whole side, i.e., 12, and takes this as the ratio between the well diameter and the well depth plus the mensor's height. That height subtracted, the remainder is the well depth.

Here there are two similar triangles, one larger, the other smaller, one in the instrument, the other in the physical figure. The larger perpendicular drops from the mensor's eye along his height and the well depth to its bottom; its base is the width, or well diameter. The hypotenuse is the sight line dropping obliquely from the mensor's eye to the far side of the bottom, and taking the entire bottom as base.

The other triangle, set up orthogonally under the same base, is similar to the first. Its perpendicular is the square side, its base the lower numerical medicline degree reading. Its hypotenuse is the mensor's sight line above its base. The smaller base and perpendicular, and their ratio are known. The larger base (the well diameter) is known, and the ratio of the larger perpendicular, through the smaller. So the larger perpendicular is known, that is, well depth plus mensor's height. Subtract the height, and you have the depth you want.

(33) Other mensors, at well edge, set at their feet a rod protruding

horizontally across the well. They adjust it back and forth until they can sight across its tip the far bottom side of the well. Then they compare the part of the rod over the mouth of the well to their own height. Using the aforementioned triangle relations, they take this as the ratio between the width of the well and its depth plus the mensor's height.

(34) Some try to determine, for a well with water at its bottom, the distance above water level. They use a rod, set transversely across the well mouth, and reflected in the water beneath, as base. They apply the same triangulation as before.

(35) Not to be overlooked is a method some use to measure the depth of a pond or stream. They fashion a copper or lead globe of very thin surface, and attach outside an easily hooked handle. Then they make a flat iron plate, longer than wide, but one width edge greater than the other. In one corner of the greater width edge they make a hook to

attach to the globe handle. On the narrower width edge, they extend a leg along the width line. The leg has greater mass at its extremity. So the length side has, at one end, a hook extending along the length direction, and, at the other, a leg extending along the width direction. They connect flat plate to globe, and sink them in the deep they want to measure. With an astrolabe, they note carefully the immersion time. As the device sinks, its hits bottom, and jarred, the globe immediately floats to the surface. With a horoscope they mark accurately the moment it surfaces, and calculate very carefully the time interval from immersion to surfacing. Next, they take a measuring rod and determine the depth, because the test must first be done in shallows, before greater depths are probed. When they have measured the depth, and found the ratio of depth to elapsed time, they can measure deeper waters (whose depth can be had directly only with great difficulty, or not at all), using this ratio and the greater or lesser elapsed time. Others compute elapsed time another way. At immersion, they put a clay bowl on the water, and mark how much fluid it collects by the time the globe surfaces. This they compare with the amount for a known depth, and apply the ratio for their tests. I myself haven't tried this, but I wanted to mention it for those who wish to do so. (Fig. 17*, Appendix B)

II. Planimetry

(36) Measurement of planar area within the visual horizon works from boundary point to observation point, or vice versa, from one section to another. It employs altimetry instruments, whose opposite squares are reversed, to read declination increase or decrease. Our discussion here will consider only the base line, and

assume that the associated perpendicular and hypotenuse are already known. We will explain, by way of example, some ideas about the process.

First, suppose one has to measure a plane figure with an astrolabe. The surveyor should stand at one end of the area to be measured, and, with astrolabe in position, depress the medicline until he can see the far end through both apertures. This done, let him compare the medicline reading to the entire square side. The ratio computed will be that of the surveyor's height to the level stretch he measures.

In planimetry, do not adjoin the surveyor's height as done in altimetry. This height is already the principal perpendicular, and both

triangles stand complete. Always be careful to match perpendicular with perpendicular, and base with base, to avoid permutation of the ratio. If, then, the perpendicular length and its ratio (to its base) are known for the first triangle, and if the perpendicular length (the surveyor's height) and its ratio are known for the second one, the base length (the line to be measured) is known. For if one of two quantities is known, as well as its ratio to the other, then the latter is also known.

(37) Here is another procedure. Put a rod of eye level height in front of you. Below the tip of the rod, at right angles to it, attach another rod as base. Then, with your eye at the tip of the first, adjust the second rod, moving it up and down, until you can sight, from the tip of the first through the tip of the second, to the end of the area you want to measure.

This gives two similar right triangles on a common hypotenuse. The larger triangle base (the unknown distance) has the same ratio to its perpendicular (the surveyor's height) as the smaller triangle base (the perpendicular horizontal rod) has to its perpendicular (the length above the attachment point). Both the smaller base length and the ratio are known; the perpendicular length is therefore known. In the same way the length of the greater perpendicular (the surveyor's height) is known along with its ratio, and of necessity, too, the length of the base from the surveyor's foot to the far end.

(38) Suppose that the rod used for a perpendicular at the end of the area is less than your own height. Step back until you can sight the far end of the area over the tip, as before. Set another rod at right angles from the tip of the first one to your position. Then compare it to that part of your height above its point of attachment. The ratio

the second rod has to that part of your height at right angles above it, is the ratio the total unknown distance will have to your entire height. The first quantity and ratio are known; so, too, the second ratio and one of its terms. The other term, therefore, is also known.

This will have to suffice now for planimetric practice; we must begin our cosmimetry study.

III. Cosmimetry

(39) Measurement of the cosmos starts at its center, evaluates diameters and circumferences of the celestial sphere and its inner orbits with the help of proportion theory, and then probes the significance of these discoveries. It delimits all cosmic objects, and sets a specific ratio and value for each distance. Our discussion, accordingly, must start at the center, and move, in fixed order, to the other parts.

Now earth is set in the middle of the physical world, where it acts as a point. It is called the global center within the circumference of a circle equidistant in every direction. Compared to the incomprehensible immensity of a celestial sphere with everything in its ambit, earth, one must admit, seems but an indivisible point, and yet relative to our own constrained ambiance, it appears exceedingly large.

So our study begins here. First, we must determine earth's size, and how to measure it.

Ancient students of natural arcana showed that earth's circumference was two hundred and fifty two thousand stades. (A stade is an eighth part of a mile, or 125 paces. So two hundred fifty two thousand stades are thirty one thousand five hundred miles.) This, divided by 360 degrees, gives seven hundred stades, i.e., eighty seven and a half miles, per degree.

History records that Eratosthenes, a famous and wise investigator of the secrets of Nature, was the first to discover this.

According to report, he proposed to estimate the extent of the earth

with an ingenious technique. He was helped by surveyors of Egyptian King Ptolemy, and from Syene to Meroe he set experienced observers and horoscopic instruments fitted with standard gnomons. He ordered that, on a given day, all should record the noontime shadow. The shadows were measured for each gnomon, and he found that no more that seven hundred stades corresponded to a unit change in gnomon shadow.

Later, he got at the truth of the matter with even greater ingenuity. At night, with an astrolabe whose circle was divided into 360 degrees according to circumference of earth and firmament, he observed the Pole Star through both apertures of its medicline, and recorded carefully the medicline degree reading. He then moved north along the meridian.

The next night, he again observed the Pole Star through the two medicline apertures. And similarly for a third and subsequent nights, until he found that the medicline reading had increased by one degree.

The next step was to measure accurately the length of his journey. This he found to be 700 stades, or 87 and a half miles. Each of the circle's 360 degrees was given this length, and with it he computed the earth's circumference as 252,000 stades, or 31,500 miles. In short, he reasoned that the 360 parts or degrees that divide Zodiac belt and celestial sphere, can be projected onto earth, and that a sector, unmeasurable and incomprehensible there, can be measured accurately here.

(40) With earth's circumference calculated, let us ask its diameter. Three and a seventh times the diameter gives the circumference. Remove a 22nd part of the circumference, and take a third of the remainder, to get the diameter. A 22nd part of 252,000 stades is 11,454 stades and twelve 22nd parts. The remainder is 240,545 stades and ten 22nd parts. Take a third of this to get earth's diameter as 80,181 and one half stades and seven 22nd parts. (Macrobius claimed 80,000 stades, or not much more.)

We should now see how ancient wisdom computed the solar altitude.

(41) At noon, when the sun is on high, its beams fall everywhere to form similar triangles. The solar altitude becomes a single perpendicular for all these triangles. But the hypotenuses are always oblique rays from sun to earth. They intercept, as bases, segments, on the surface of the earth below, between themselves

and the common perpendicular of the triangles. An object, set erect as perpendicular, casts a shadow, under the sun's rays, away from the sun, and forms a similar triangle, since it is a right triangle on the same hypotenuse. Consider the ratio between the shadow on the base and the perpendicular object casting the shadow. This will surely be the ratio between the segment, extended as a triangle base from the shadow tip back to the foot of the solar altitude, and the solar altitude itself.

The Egyptians get credit as the first to compute solar altitude this way. And, indeed, because of flat land and the sun's proximity, they were readily able to measure the interval distance.

The height of sun above earth, then, was found to be four million, eight hundred and 20 thousand stades, or forty eight hundred thousands plus 20 thousands. Consider an equal length from another point to the sun's orbit. The double length is ninety six hundred and 40 thousands. Add to this earth's diameter (eighty thousand, 181 and one half stades

plus seven 22nds) to get the total solar orbit diameter from any orbit point to an antipodal one. It is ninety seven hundred and 20 thousands, 181 and one half plus seven 22nds. Now, according to circular geometry, add a seventh part of this to its triple, to get thirty thousand thousands, five hundred thousand, 49,000 plus 142 and five sixths plus a seventh of a sixth part. This is the precise length of the solar orbit.

Why Macrobius neglected this computation I do not know. Perhaps natural philosophers do not admit the diameter of the center of the cosmos as an indivisible point in measurement. But let us summarize. Earth's diameter is eighty thousand, 181 and one half stades plus seven 22nds, and its circumference two hundred fifty two thousand stades. The solar altitude is forty eight hundred thousands plus twenty thousand stades. The solar orbit diameter is ninety seven hundred and twenty thousands, 181 and one half stades plus seven 22nds, and the solar orbit itself thirty thousand thousands, five hundred thousand, 49,000 plus 142 and five sixths plus a seventh of five sixths.

(42) Before we start another topic, we should explain how someone anywhere far outside Egypt can calculate the solar altitude.

I shall propose two ways; one involves a gnomon shadow, the other, a medicline and solar rays.

Here is the gnomon method. Set up, perpendicular to the ground, a gnomon of arbitrary length to catch the sun's beam at noon. Compare carefully the shadow the gnomon casts with the gnomon itself. Put this as the ratio between solar altitude and the segment it intercepts. Now the length of this segment cannot be measured

because of its immense size, so it must be gotten indirectly. For this, let the observer move back to a second station. Again, with the same gnomon set up, let him record the shadow for comparison purposes, and once again take the ratio found between gnomon and shadow as that between solar altitude and segment intercepted. He should know that the excess of the second base over the first is just the distance between his first and second stations. This carefully measured, everything else can be computed at once from the data. We may omit the details, which have already been explained.

(43) Another method, however, is more accessible and less labored. A difficult and deep problem can, surprisingly, be resolved without complex argumentation. The circumference of the entire firmament is divided into 360 degrees, and surrounds the circumference of earth, fixed at its center. So lines can be drawn from points of division in the firmament down to earth, and the circumference of the terrestrial globe broken into

an equal number of sections. In this way, parts of the celestial sphere, though of incomprehensible magnitude, can be paired through circular degree measure with carefully measured sections of earth's circumference.

Consider, then, that from any two degree markings of the 360 degree marks in the firmament, straight lines are drawn to earth's circumference. Precisely as many degrees as are contained between the two chosen firmament markers will be included between the lesser degree marks on earth's circumference. The ratio the firmament interval has to the total firmament circumference will surely be the ratio the terrestrial segment has to the total terrestrial circumference.

Suppose, now, the sun at the meridian. With astrolabe raised, fix the medicline against its beam. When we observe, through the two medicline apertures, that the sun is directly crossing the meridian, we check the instrument at once and record the number of degrees between medicline and meridian. This calculated, we can determine how many of the 360 firmament degrees lie between our zenith and the sun.

From this, too, we can set precisely that many of the 360 firmament degrees between ourselves and the place where the sun is directly overhead. The length of one three hundred sixtieth part of earth's circumference is known. So the total length of all three hundred sixty degrees is also known. If, too, the length of the basis and the ratio of the perpendicular to it are known, the length of the perpendicular is known, in accordance with the rule already given.

Enough for the moment about solar altitude. We must now tell how

to calculate the size of the solar sphere and its diameter.

(44) At equinox, before sunrise, the cosmimeter should set up a horoscopic device marked clearly with hour interstices, and its gnomon. These should be arranged so that the pole, catching the first ray of sunlight, casts a shadow, in the other direction, on the initial boundary line of the first day hour. The exact spot to catch the first shadow of the rising sun is marked accurately. The shadow trace is watched closely until the entire solar disk has emerged above the horizon, and seems to rest upon it. Then the spot marked by the pole shadow is also noted.

Let him record the difference between first and second shadow observations, which measure the total solar disk between its edges, i.e., its diameter. This is found to be a ninth of the space between the initial and terminal boundaries of the first daylight hour. It follows that a single equinoctial hour is equivalent to nine complete sunrise times. The

celestial hemisphere is rotated in 12 integral equinoctial hours, the entire celestial sphere in 24. Nine multiplied by 24 is 216. Consequently, the solar disk is one two hundred sixteenth part of its orbit.

Some may question this argument because of the sun's motion, in that the sun, moving along its orbit with no more delay than its bulk demands, seems, at its rising, to delude the observer by pausing. But one can, with an astrolabe, do a very precise experiment, even as the solar disk arises and momentarily rests on the horizon. Focus the medicline at the solar apex. Compare the lower degree reading above the medicline with the whole quadrant. This gives the ratio between the sun's diameter and the full quadrant.

(45) Once we have found the sun's diameter, circle geometry gives its circumference. According to rule, a seventh part must be added to triple the diameter. But the sun's diameter, known by precise argument as one two hundred sixteenth part of its orbit length, is found much larger than earth's diameter, yet by no means twice as large.

The solar diameter falls short of twice the earth's diameter by 18,932 and a sixth stades and half a sixth. (Macrobius said it was almost twice that of earth. Here his claim was as much exaggerated as his earlier estimate of earth's diameter was deficient. Perhaps he wanted to spare the reader tedious numerical computations.)

We now use the sun's diameter and the solar orbit diameter to estimate the length of earth's shadow in terms of its diameter. Let us assemble the data to see the implications. The solar orbit

diameter, as noted, is 9,720,181 and 1/2 stades and seven 22nds of a stade. The sun's diameter is 141,431 stades plus a sixth, a twenty fourth and a two hundred sixteenth part of one plus five sixths plus a seventh of a sixth part of a stade. This is the sun's entire diameter. Earth's diameter is 80,181 and 1/2 stades and seven 22nds.

(46) Consider two perpendiculars set up on the same base. The one at the end is twice the other; the second, set at midpoint, half the first. Both meet the same hypotenuse to their common base. If they are not in double ratio, different hypotenuses come from end and midpoint perpendiculars to the common base. In fact, the hypotenuse from the deficient perpendicular, whether at middle or end, then lies beneath the higher hypotenuse of the greater perpendicular.

Here are examples: first, two perpendiculars, one double the other, erected on the same base at end and at midpoint. Next, two perpendiculars on the same base, at end and at midpoint, this time one

not double the other, and the end point one greater than double the middle one; alternately, the end point perpendicular less than double the midpoint perpendicular. Figures illustrate the cases. (Figs. 18a, 18b, 18c)

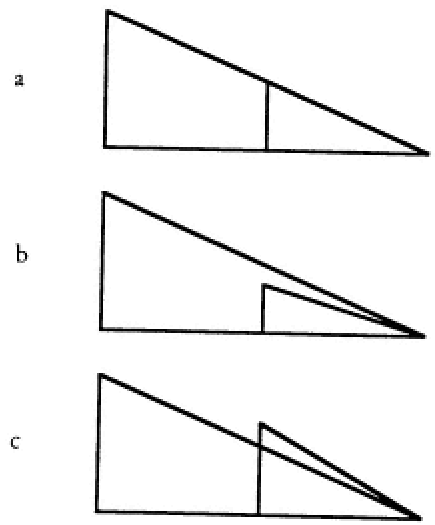


Figure 18

In the latter case, if the amount by which the first perpendicular is less than twice the second be adjoined four times to the base (or as often as necessary for proportionality), then the hypotenuses will coincide, and so both perpendiculars, meeting a common hypotenuse, will share a common augmented base.

(47) Let the solar orbit's diameter act as a base, going from one circumferential point, through the orbit center, to a second. Let it cut earth's diameter at its center, leaving half above and half below. As a result, the solar disk is attached, at its center, to the point where the solar orbit diameter intersects its own circumference, so that upper and lower perpendiculars are seen erected on it as base: at the first end point, the sun's diameter, half above and half below, and at midpoint, earth's diameter, half above and half below.

So if the solar disk's total diameter were double that of earth, half the perpendicular erected about the first base endpoint would be double half the perpendicular erected about the base midpoint. But now because the entire diameter is less than double the midpoint diameter, it necessarily follows that half of this is less than double half of the other. Indeed, any quantity that exceeds another by some difference, necessarily has its half exceed the other's by half their difference.

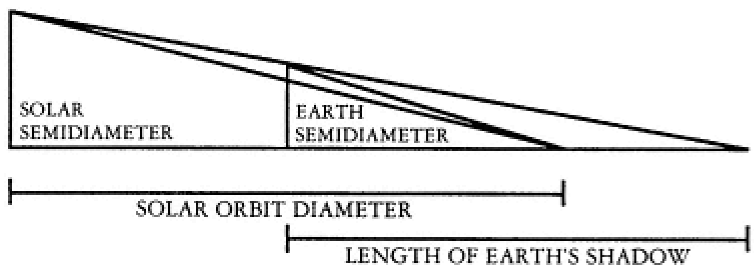


Figure 19*

We have already seen that the sun's diameter is one hundred 40 one thousand, four hundred thirty one stades plus a sixth, a twenty-fourth, and a two hundred sixteenth part of one plus five sixths plus a seventh of a sixth part of a stade. The earth's diameter, as we have said, is eighty thousand three hundred thirty three and a third stades.

Subtract a smaller number from a larger one to get their difference. Subtract eighty thousand three hundred thirty three and one third from the larger diameter number to get sixty thousand eight hundred ninety plus eleven twelfths, a twenty- fourth, and a two hundred sixteenth part of a unit plus a twenty- fourth plus a seventh of a twenty-fourth part.

This is the difference between the diameters of sun and earth. Half

of this is 30 three thousand four hundred forty five plus five twelfths plus a twenty-fourth plus a forty-eighth plus a four hundred thirty second of a unit plus a twenty-fourth. By this much does half the sun's diameter exceed half the earth's. Because this difference does not add on to make a double ratio, the perpendiculars erected on the common base do not have coincident hypotenuses.

(48) The hypotenuses will coincide provided the amount by which the double of the second perpendicular exceeds the first be added four times to the base.

Take half the solar diameter, i.e., the first perpendicular, which is 70,615 plus one and a half twelfths, a forty-eighth, a four hundred thirty second part of a unit plus a twenty-fourth plus a seventh of a twenty-fourth. Subtract this from double the second perpendicular (earth's diameter; i.e., eighty thousand three hundred thirty three and a third) to get a remainder of seven hundred eighteen and a third.

Add the remainder four times to the base, i.e., to the solar orbit diameter, for a total addition of two thousand eight hundred seventy three and one third stades. This is how far earth's shadow extends beyond the solar orbit diameter. (Macrobius said it extended only out to the sun's orbit.)

Our text figure shows earth's sphere and its diameter, and outside it, the solar orbit and its diameter. The solar disk and its diameter have been attached to the orbit diameter extremity, and hypotenuses drawn from each perpendicular. (Fig. 19*)

(49) Accordingly, in the figure, the hypotenuse drawn from the first perpendicular (the solar diameter extremity) cuts under the top of the second perpendicular, so that it meets, in the common base, the hypotenuse coming from the top of the second perpendicular (earth's diameter), at the spot where the circular orbit diameter intersects the orbit. This proves the solar ray cannot cut off earth's shadow there, because earth's greater diameter intervenes to make it impossible.

But suppose the base is extended to the spot that the sun's ray, descending along the hypotenuse across the top of earth's diameter, hits. This spot, necessarily, is the end of earth's shadow (cast away

from the sun from either end of earth's diameter). Here, too, there is a sharp boundary for light and shadow, so that beyond it light does not admit the presence of shadow, but alone abounds in the region of the sun.

(50) As circumstances allowed, we have studied earth's diameter and circumference, the solar altitude, the diameter and length of the solar orbit, then the diameter of the solar sphere, and the length of earth's shadow. Now let us study what Nature allows about the horizon circle and those objects within it that cast a shadow. But first, some preparatory remarks.

(51) Consider two circles, one circumscribed by the other. Suppose that the circumscript circle is all light inside, but its exterior entirely dark. Suppose a part of its circumference opened. Then emergent light intercepts the larger circumference to the degree that it emerges, in this way, from the smaller one. (Fig. 20*)

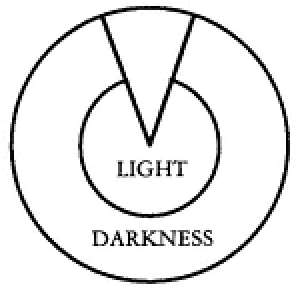


Figure 20*

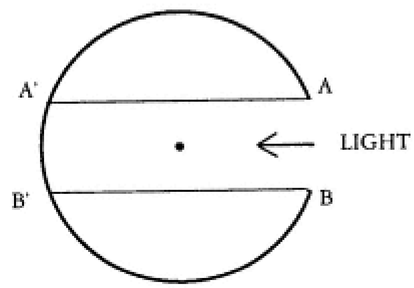


Figure 21*

(52) Consider a disk entirely dark inside, but outside light everywhere. Suppose an opening made in the disk's rim. Incoming light then intercepts precisely the same part of the rim wall opposite as it does of the entrance exterior. (Fig. 21*)

(53) Consider now two circles, one circumscribing the other. Suppose the circumscribing circle entirely dark inside, but outside light everywhere. Suppose an opening at some point of the outer rim, so that light entering hits the inner circle from without. Inside the light hits a circle smaller than the exterior one, and does so from without, so it falls upon a greater proportion of the smaller circle, though entering by equal space and interval. A smaller disk within has a greater proportion intercepted than either of the first two, a larger disk, a lesser proportion, and this whether light emerges from within, or enters from without. (Fig. 22*)

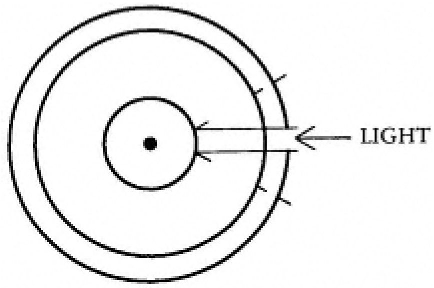


Figure 22*

(54) Suppose one circle drawn about another any distance within. Light rays from an arbitrary point of the exterior circle that fall on either side to make tangential contact with the interior circle intercept a part of it, greater or lesser according to the separation distance of the circles: the greater the separation, the greater the intercepted part, the lesser the separation, the lesser the part. Rays that pass beyond the inner circle contact points to the outer rim intercept some part of it from either side. What the two intercept on the farther rim is greater than what they intercept on the inner circle. (Fig. 23*)

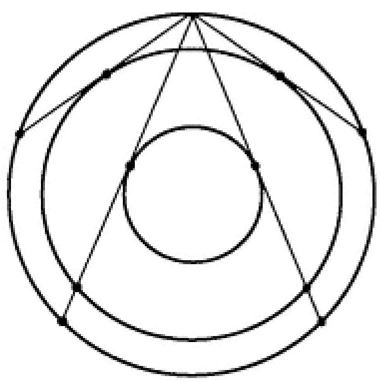
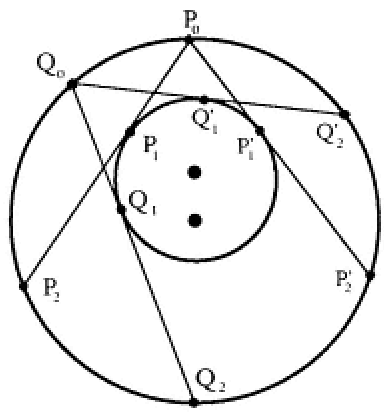


Figure 23*



$P_1 P_2 = P'_1 P'_2;\ Q_1 Q_2 > Q'_1 Q'_2$

Figure 24*

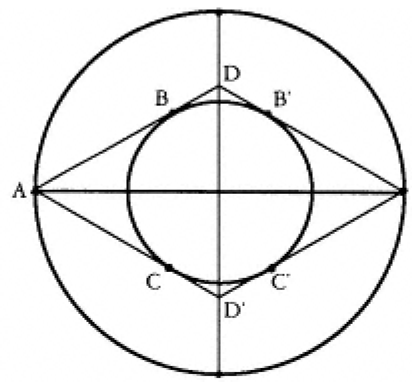
Depending on location of the first point, the contacts between inner circle and far intersection point are sometimes equal, but sometimes not, greater to this one, less to that. But a common equal contact length is found for a location that is always unique for the two circles, and then contact is equidistant between tangent point and far rim point and the second tangent point and corresponding far rim point. (Fig.24*)

The inner circumference part between contact points forms the diameter of the horizon.

(55) Imagine several concentric circles of various diameters. Intersection with the outer circle occurs less. Only the inner circle is contacted twice, the outer circle, and no other, three times, but all intermediate ones, however many, four times.

Suppose several concentric circles, and that from an arbitrary point of the circle second from the center, rays, unbounded, emerge to touch the first circle. They intersect twice exteriorly circumscribed circles. And, in general, every circle, from which tangent lines

emerge, has to be met three times, the circle to which the tangents proceed, twice, circles intermediate to these two, four times, exterior circles twice, and circles interior to both, not at all. (Our terms: contact, touch, cut, intersect, are loosely used.)



$\triangle ABC = \triangle A'B'C' > \triangle BDB' = \triangle CD'C'$

Figure 25*

(56) Consider two antipodal points on a circle. From them, draw tangent lines to an inner circle, so that these tangents, intersecting on both sides of the inner circle, form four triangles. Sometimes all are equal, sometimes an opposite pair larger and the alternate opposite pair smaller. Tangent line intersection occurs at the mid-line of the interior circumference, and on either side of it are two opposite triangles. When these triangles are extended ever greater in height, tangency does not move beyond the mid-line, nor can any more of the interior circle be intercepted, and what is below the mid line, must necessarily be out of sight all the time. This holds for four equal triangles, or two greater and two smaller triangles, or two maximal triangles. (Fig. 25*)

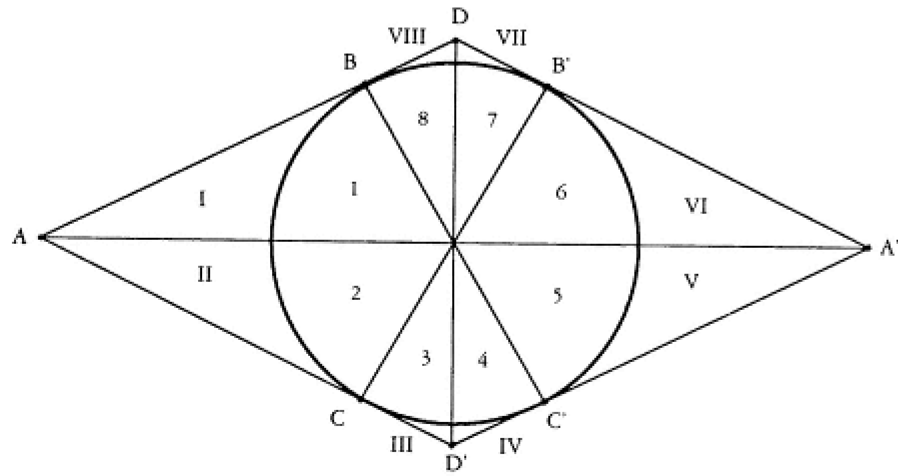


Figure 26*

From the two points whence come the tangent lines, draw two straight lines to the center. Draw, too, straight lines from all contact points to the center. Eight triangles are formed between the exterior and interior circles, and eight other triangles between interior circle and center. Sometimes all are equal, other times not, as in the figure. (Fig. 26*)

With this, let us set the horizon and the content of its periphery, because the earth's globe, by reason of the immense sweep of its circumference, approximates a plane, and unfolds itself to a vision extended for greater comprehension.

(57) Let vision be a ray from a (circumference) point, going out directly to encompass a circumference on both sides. Anything that comes from an exterior circumference intercepts the (interior) circumference on both sides with lines coming from its source to contact points. What it apprehends is sometimes smaller, sometimes greater, depending on elevation. That apprehended from a lesser one is greater relative to it, and that from a greater one, always smaller relative to it. The outer circumference is

apprehended inside and lower along the vision line. The interior circumference is apprehended higher. Yet, what of the former is apprehended lower, and of the latter higher, coincide in one visual line, as is always true.

A length that reaches down below contracts vision into one line, as is the case here.

From this, Macrobius says that vision fails beyond 180 stades, and that, on encountering earth's rotundity, forms a curved horizon. Thus the horizon circle cannot be more than 360 stades wide, a distance of 180 stades from its center to any part. This he said because he did not add or subtract anything (which the computation demands) to account for discrepancy in human height. Perhaps he allowed diverse stades and paces for different heights.

More details about the horizon we leave to a subsequent book on parallels, colures, and other celestial circles.

Appendices

Appendix A
Two Station Altimeter Methods

1. Far distant heights. The surveyor gets the ratio A = a/b' from the astrolabe reading at S', and the ratio B = a/b" from that at S". He measures the distance b"-b' directly, although neither b' nor b" can be so measured. Since b'= a/A, and b"= a/B, it follows that b"-b' = a(1/B-1/A), so the height (altitudo) a= (b"-b')/(1/B-1/A).

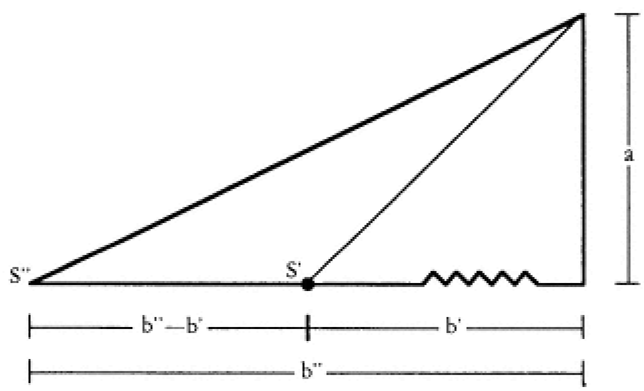


Figure A-1

2. Nearby heights. The surveyor gets the ratio A=(a-m)/b' from the astrolabe at S', and the ratio B=(a-m)/b" at S". He measures directly the distance b"-b' between S' and S". Then height a=(b"-b')/(1/B-1/A) + m. Thus the surveyor's height m is added directly to the previous result. For distant heights, m is inconsequential relative to a and b'.

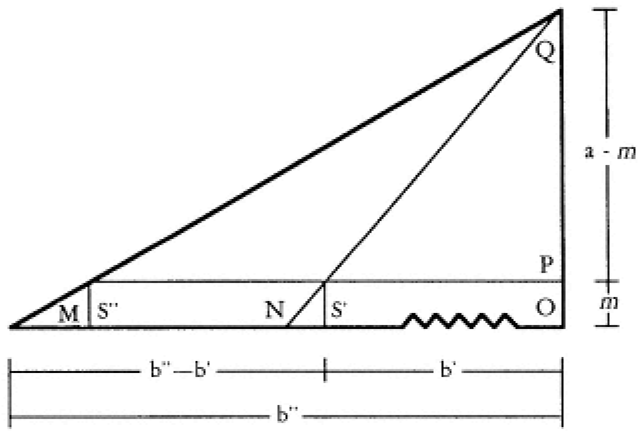


Figure A-2

To understand how to "add the surveyor's height proportionally," and the warning of #18, consider a third calculation. Again, A is the astrolabe ratio computed at the first station S', B the ratio computed at S''. Let OP = m, CO = a, MS'' = m/B, NS' = m/A, OS' = b', and OS'' = b''. Then we have:

$$A = (a - m)/b' = a/(b' + m/A),$$

$$B = (a - m)/b'' = a/(b'' + m/B),$$

so that

$$a/B = b'' + m/B \,;\, a/A = b' + m/A,$$

and hence

$$(b'' + m/B) - (b' + m/A) = a(1/B - 1/A),$$

or

$$a = [(b'' + m/B) - (b' + m/A)]/[1/B - 1/A].$$

Here the "surveyor's height adjustment" at the first station is m/B, and at the second m/A, so that the true difference between bases is

$$(b'' + m/B) - (b' + m/A).$$

One can readily show that the two expressions derived here for the height a are equivalent. The first adds the surveyor's height directly; the second has the surveyor's height added in the appropriate proportion to each station base before the difference in the adjusted bases is computed.

Heteroplane variants (#31). Suppose the mensor at C wants to measure the height a of GF. The procedure to determine a has three parts (Fig. A-3). The height h of the sheer cliff DE, the height v of the mensor's vertical rod AC, and the length d of his distance CD from the edge of the cliff are all known. The first step is to use the similar triangles ACD and DEF to compute the length b of EF, the distance from the foot of the cliff to the foot of the object. The second step uses (1/2)v, d, and h in the similar triangles BCD and DEH to compute b+b', the length of EH, and so B'=(b + b')-b', the length of FH. The third step uses h, b+b', and b' in the similar triangles EDH and GFH to compute a, the height of GF.

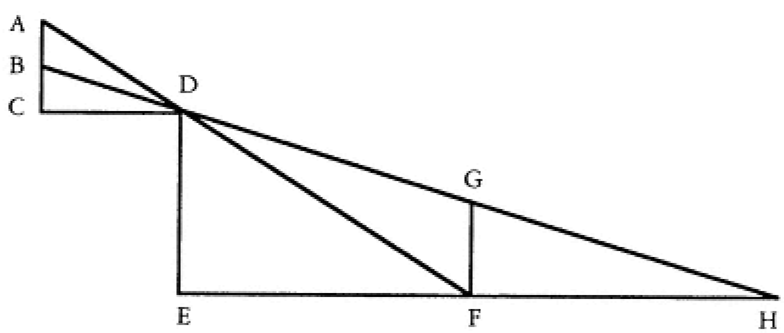


Figure A-3

These methods, found in the *practica geometriae* tradition of Gerbert, Hugh, and their successors, can be traced to the time of Heron of Alexandria (1st century A.D.). His treatises reported

traditional methods for solving practical problems, and show points of contact with the Roman agrimensores. Heron's *Dioptra* describes this surveyor's instrument, and then treats problems of heights and distances, engineering projects, and mensuration problems. In particular, he has

methods for the distance between two inaccessible points, for the height of an inaccessible point, and for the difference between the height of two inaccessible points.

"Two station" methods also appear in early Chinese and Sanskrit texts. The Chinese geometer Liu Hui (3rd century A.D.) told how to find the height of a sea island and its distance from land with the use of two poles of equal height. A second problem in his treatise *Island in the Sea* proposed to measure the height of a tree atop a mountain and a two pole method was described. The extent of Greek influence on Liu Hui is a debated point (van der Waerden 1983, p. 193).

The Indian mathematician Aryabhata (6th century A.D.) also described, succinctly, "two station" methods in verses 15 and 16 of Ganitapada, the second part of the extant Sanskrit text *Aryabhatiya* (Sarasvati Amma 1979, chapter X). Here, too, the extent of Greek influence is debated.

Appendix B
A Mechanical Bathometer

The diagram of Figure 17* interprets the PG text (#35) otherwise than the scribal manuscript diagram Baron prints as Figure 17 (no. 46 <<sic>), which, seemingly, both misreads the text and makes the scheme unworkable. When the device pictured here strikes the river bottom (*fundum*), the mass (*crassitudo*) at the foot (*pes*) of the leg extension tips the device downward through the angle A. Because angle A is made greater than angle B of the hook (*uncus*) by which the iron plate (*plana formula*) is attached to the handle (*ansula*) of the float (*globus*), the float can slip free of the hook and rise to the surface when the plate tilts downward.

To calibrate the time readings, we suppose that

$$v'' = R\,v',$$

where v' = velocity of descent, v'' = velocity of ascent, and R is a constant. Then, if d = unknown depth, and T = total time for descent and ascent of the float, we get that

$$T = (d/v') + (d/v'') = (d/v')(1/R + 1),$$

So,

$$d = (Rv'/(1+R))T = C\,T, \text{ where } C = Rv'/(1+R),$$

Thus, if the time T' for a known depth d' is observed, we can immediately determine the value of the constant C as d'/T'. With this fixed value of C, the relation

$$d = C T$$

gives the unknown depth d in terms of the observed time T for the float to sink to the bottom and rise again. The device will work in theory, provided there are accurate ways to determine the time T. In practice, success seems less assured.

Alberti described this device in *Ludi Matematici* where he attributed it to "*scrittori antichi.*" Vagnetti thought he might refer to Savasorda (Abraham bar Hiyya), and suggested that a description could be found in Plato of Tivoli's 1145 (?) Latin translation, *Liber Embadorum* (Vagnetti, 1972, p. 207). A reading of M. Curtze's critical text and German translation of the *Liber Embadorum* failed to locate this reference

(Curtze, 1902). But Millas y Vallicrosa has printed an excerpt from the little-known tenth century Latin ms. 25 in the library at Sancta Maria Monastery of Ripoll that does describe the device, and cited another in the Bibliotheque National de Paris (ms. 11248 F. L.) with a variant text and a small sketch of the device (Millas y Vallicrosa, 1949, p. 71). The anymous Ripoll ms. is one of several witnesses to Arabic astrolabe texts. An eleventh century British Museum maunscript (Old Royal 15 B IX) also presents the depth measurement device to illustrate use of an astrolabe with cursor to calculate time intervals. Such mss. witness to sources that Hugh, and later Alberti, could draw upon.

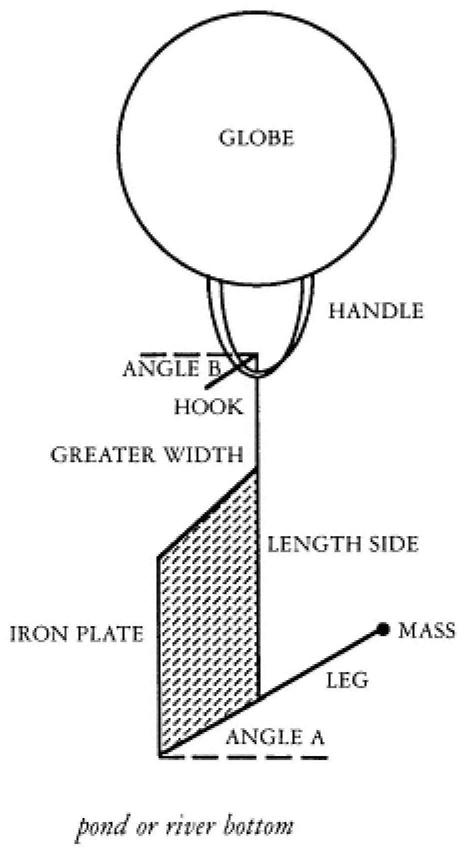


Figure 17*

Appendix C
A Text of Geometria Gerberti

We have poured over geometric diagrams in an exhausting study of scientific discoveries. To avert collapse from utter fatigue, let's refresh the mind with some military exercises. In place of the usual daily fare, the body is refreshed by new and different treats. Just so, in place of onerous scientific austerities, the mind is refreshed with a bit of poetic fancy. We can dust some mental cobwebs with a "military application".

Suppose you want to find an unknown height. Here's a shooting good way to do it. Get a bow, arrow, and line. One end of the line is attached to the tip of the arrow. Keep the other end in your hand. Shoot the arrow from the bow so that it hits the apex of the unknown height. Then tie the end of another line in the same way to an arrow or javelin. Throw either one, as you please, so that it hits the foot of the height, just as the first hit the apex. This done, pull back both lines. Measure carefully both lengths, and multiply each length by itself. When each product has been calculated, subtract the smaller from the larger. Then compute carefully the side of the remaining square number. This process gives you the precise height in feet or cubits as the side of the square.

To explain. The diagram shows the height and the lines as marked. The height to be measured is AB. Suppose the length of the first line AC (to the apex) is five units, and the length of the other line BC (to the foot of the height) four units. Thus the square of the first number is 25, the square of the second is 16. Sixteen subtracted

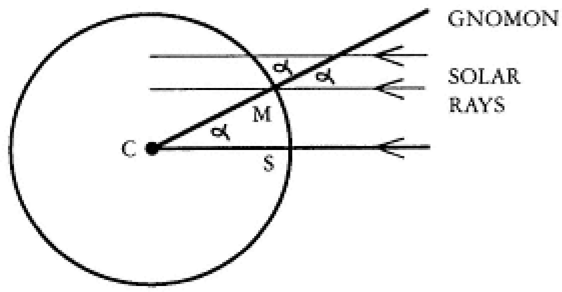
from 25 leaves 9, a square number whose side is 3 (for 3 squared is 9). So the height AB is 3. But it can happen that the side of the square is not an integer. The computations are then more difficult, and for lack of space, we omit a discussion.

(Bubnov: *Gerberti opera mathematica*, *Geometria incerti auctoris*, III, 26, pp. 334-35.)

Artis Cuiuslibet Consummatio II, 23 (Victor 1979, p. 308) lists the bow and arrow method as a practical technique, and gives the 3-4-5 triangle as an example, but says nothing about irrational square roots. Whether the ACC author knew directly this text is problematic (Victor 1979, p. 83).

Appendix D
The Methods of Eratosthenes

According to the relatively complete, but second hand account of Cleomedes (written probably about the middle of the first century B.C.) Eratosthenes (276-194) assumed the surfs rays parallel as they strike the spherical earth (Fig. D-1). At Syene S (Aswan in Upper Egypt), the sun is directly overhead at the summer solstice noontime. At Meroe M (Alexandria), a gnomon casts a shadow, making an angle of 1/50th of a circle with the solar rays. Assuming that Meroe was exactly 5000 stades directly north of Syene, Eratosthenes calculated the earth's circumference as 250,000 stades. This was adjusted to 252,000 stades, perhaps for divisibility reasons and ease in computation. See J. L. E. Dreyer, *A History of Astronomy*, pp. 174-76, and O. A. W. Dilke, *Mathematics and Measurement*, pp. 35-6, for a discussion of the length of a stade. I. Thomas, *Selections Illustrating the History of Greek Mathematics*, Vol. 2, prints the text and translation of Cleomedes who recorded Eratosthenes's figure as 250,000 stades, and a text excerpted from Heron's *Dioptra* that recorded 252,000 stades. Vitruvius (*De architectura*, I, 6, 9) also recorded 252,000 stades as Eratosthenes' figure, along with the remark (I, 6, 11) that "some claim that he could not have determined the true measure of the earth." Stahl notes the contemporary agreement of scholars that "Macrobius and (Martianus) Capella were the authorities responsible for the wide adoption of Eratosthenes's figure of 252,000 stades for the earth's circumference in the Middle Ages." (Stahl 1952, p. 51.)


GNOMON
SOLAR
RAYS
C
M
S
α
α
α

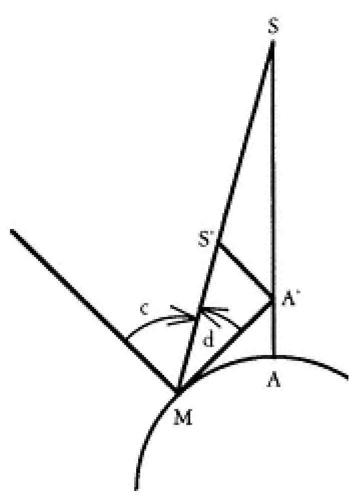
We may note that, as Baron's historical apparatus indicates, the PG text of the gnomon shadow method attributed to Eratosthenes is found almost verbatim in *Geometria incerti auctoris IV*, 60 (Bubnov 1899, pp. 362-63). Bubnov cites Martianus Capella's *De nuptiis philologiae et mercurii* (VI, #595-98) as a reference for this, but remarks that the *incertus auctor* seems to have based his text not on Martianus, but rather on one of the *Gromatici Veteres* (the Roman surveyors), whose manuscript he had at hand.

Unidentified as yet is the text source of the second method given in PG (#40), and there attributed to Eratosthenes, which exploited the change in declination of the Pole (Star) as one travels along the meridian. The method is, in essence, that of Posidonius who, a century after Eratosthenes, observed the declination change of the star Canopus from Meroe (Alexandria) along the meridian to Rhodes, and estimated the distance, to compute earth's circumference, as we read in Cleomedes. Perhaps he did so, as Dreyer suggests (*History of Astronomy*, p. 177), to test Eratosthenes' result by a different method, not using the sun with the attendant difficulty of observing sharply the end of a shadow. Posidonius had a good idea; his numerical data, and resultant value for earth's circumference, were quite erroneous. (Nevertheless, Ptolemy adopted Posidonius' estimate at face value in the *Almagest*.) But how his method found its way into the early medieval tradition and PG (which puts Eratosthenes to using it with the Pole Star in place of Canopus) is not clear. Neither Macrobius or Gerbert know the procedure, nor did Martianus Capella, whose badly confused attempt to explain the acknowledged method of Eratosthenes (*De nuptiis VI*, #597) contrasts with PG's succinct and accurate explanations for both cases. I. G. Kidd's commentary

evaluates witness of the Posidonius-Cleomedes text transmission (Kidd, 1988).

Appendix E
An Erroneous Solar Altitude Argument

The argument of PG #43 seems to be this. At noon, the mensor observes through the astrolabe the angle d of the sun's elevation, and so the angle c = 90 - d (Fig. E-1). Reversing the procedure of Eratosthenes, he can compute with angle c the arc distance along the meridian to the place where the sun is directly overhead. This arc distance is taken as the length of the base of a right triangle with angles c and d. The sun's height (*altitudo solis*) is identified with the perpendicular (*cathetus*) of the triangle, which is readily computed. Thus, in the figure, Hugh wants to compute distance AS (*altitudo solis*). He observes angle d, computes c, and with it arc distance MA as (70 stades/degree) x (c degrees). He sets up right triangle MA'S' with known base MA' (= MA) and angles c and d to compute A'S', identified with AS, an immense error.



Appendix F
A Geometric Text from *De arca Noe morali*

[This text, omitted in the translation done for *Hugh of Saint Victor: Selected Spiritual Writings* (London, 1962), has been made, with corrections, from the text printed in Migne, *Patrologia latina*, clxxvi, 628C-629D.]

The great discipline of geometry can help us probe details of the Ark's dimensions, and here we may note a few items. The smallest field measure length given a name is the digit. Anything less than a digit is a part, that is, a half, third, or fourth of a digit. To measure with digits, put two thumbs side by side, and let your fingernail roots trace a straight line. Four digits make a palm, four palms, a foot. A foot and a half make a minor cubit, six minor cubits, a major one. A major cubit has nine feet, a minor cubit one and a half. Accordingly, a length of three hundred major cubits has one thousand eight hundred minor cubits, or two thousand seven hundred feet, but ten thousand eight hundred palms, and forty two thousand two hundred digits.

Also, five feet make one pace, one hundred and twenty five paces make one stade, and eight stades, one mile. Thus the length of Noah's Ark was five hundred forty paces, or four stades (half a mile) and forty paces. In the same way, we can get numbers for its width and height.

To compute the area in squares, whether feet, or cubits, multiply the greater side by the lesser; the result will be the area. For

example, fifty times three hundred is fifteen thousand, the area in square cubits.

Here is the rule for the diagonal line of a rectangle. Take each side and multiply it by itself. Add these together, and find the side of the resultant square sum. This is the diagonal line.

For example, three hundred times three hundred is ninety thousand. Fifty times fifty is two thousand five hundred. The side of this sum is three hundred and four and a fraction (that is, not an integer). Half of this is one hundred and fifty two and a lesser fraction, a segment extended from a corner to the midpoint. Take it as a base. In the center of the Ark, erect a perpendicular to the height of thirty cubits, and along the perpendicular itself put fifteen cubits above the third story. Make this a perpendicular, and at its foot set out this base line according to geometric rule. You will see that, remarkably, all hypotenuses from the

corners measure one hundred and fifty two and a fraction. If, also, you join to the perpendicular a base line coming from a face, you will get hypotenuses coming from the face extended one hundred and fifty two units and a fraction. But those coming from the sides are twenty eight units and a fraction, with twenty five cubit bases.

To find the hypotenuse, multiply the perpendicular into itself, and similarly, the base, and add the results. Take as hypotenuse the side of the number that arises in this way.

In the Ark, of course, wooden beams stretched obliquely from bottom to top replace the hypotenuse. The perpendicular is a column in the middle of the Ark raised to the top cubit. The base is a line reaching from one corner of the Ark to the perpendicular. The diagonal line reaches from one corner to its opposite. There are many other subtle geometric relations found in these triangles and rectangles that we omit with a certain reluctance.

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